TEACHING MATHEMATICS: RETROSPECTIVE AND PERSPECTIVES

Proceedings of the 10th International Conference

Edited by Madis Lepik

Tallinn, May 14-16, 2009

TEACHING MATHEMATICS: RETROSPECTIVE AND PERSPECTIVES,

Proceedings of the 10th International Conference, Tallinn University, May 14-16, 2009.

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Conference was organised by

Institute of Mathematics and Natural Sciences, Tallinn University

Conference was supported by

European Social Fund Program *Eduko*

ISBN 978-9985-58-688-4

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TEACHING AND LEARNING MATHEMATICS: RESEARCH PAPERS

PREFACE

The 10th conference *Teaching Mathematics: Retrospective and Perspectives* was held at Tallinn University, Estonia with around 70 participants from seven countries.

These conferences have traditionally been a meeting place for Baltic and Nordic researchers, teacher trainers and others interested in mathematics education research and development. Scientific cooperation of researchers and educationalists from the three Baltic countries, Estonia, Latvia and Lithuania, has a long history. The first joint seminar on mathematics education research and developmental work took place in 1984 in Liepaja, Latvia. Since 1998 the seminars have taken the form of annual conferences. The list of seminars and conferences is as follows:

Seminars		Conferences	
1984	Liepaja, Latvia	1998	Šiauliai, Lithuania
1985	Tartu, Estonia	1999	Riga, Latvia
1986	Vilnius, Lithuania	2001	Liepaja, Latvia
1988	Riga, Latvia	2003	Tallinn, Estonia
1990	Tartu, Estonia	2004	Liepaja, Latvia
1992	Vilnius, Lithuania	2005	Vilnius, Lithuania
1993	Daugavpils, Latvia	2006	Tartu, Estonia
1994	Tallinn, Estonia	2007	Riga, Latvia
1995	Šiauliai, Lithuania	2008	Vilnius, Lithuania
1996	Liepaja, Latvia	2009	Tallinn, Estonia
1997	Tartu, Estonia		

The aim of this conference was to present and discuss recent research and developmental work, and to share teacher training experiences.

Conference themes were:
- Teaching and learning mathematics
- Education and professional development of mathematics teachers
- Technology in mathematics education
- Teaching and learning mathematics at the tertiary level

Plenary sessions were intended to map the landscape of mathematics education research and development in the Baltic and Nordic countries. Markku Hannula and Barbro Grevholm in their lectures outlined the broader picture, describing the state of the art of mathematics education research internationally and in the Nordic countries. More detailed overviews of research activities and doctoral education in Estonia, Latvia and Lithuania followed. Also the major developments and most important trends in mathematics education practices in the Nordic and Baltic countries were described in the plenaries.

This volume contains a selection of plenary lectures and papers presented during the conference sessions.

MATHEMATICS EDUCATION RESEARCH IN THE NORDIC AND BALTIC AREA

INTERNATIONAL TRENDS IN MATHEMATICS EDUCATION RESEARCH

Markku S. Hannula, University of Helsinki

Abstract

This article gives an overview of international trends in mathematics education research. This overview is based mainly on a recent Handbook on PME activities 1976-2006 and PME conference proceedings 1997-2007.

Introduction

In this article I try to give an overview of the different trends in mathematics education research worldwide. This overview is mainly based on activities of the International Group for the Psychology of Mathematics Education (PME).

I have chosen PME as the main lens for this overview because it is the organization whose activities I have participated most actively throughout my own research career, and because they have recently published a review of the PME research activities for their first 30-years of existence 1976-2006 (Gutiérrez & Boero, 2006). Moreover, yearly PME organizers attract a large number of participants throughout the world and their conference proceedings provide a classification of research topic for each accepted research report. Readily on my shelf were proceedings for the PME conferences for years 1997, 1998, 2000-2003, and 2005-2007.

In addition to the review of aforementioned volumes, also my personal reflections have guided this overview.

Trends in theoretical frameworks

The early years of PME research were characterized by a strong emphasis on cognitive psychology. Constructivism ascended as the theoretical framework over the years 1985-1995 and it became the dominating framework. However, constructivism has been too general a framework, and several bridging theories have been developed that connect the constructivist principles with specifics of mathematics learning. For example APOS (Dubinsky& Mcdonald, 2001) and theory of didactical situations (Brusseau, 1997) are such bridging theories.

After the cognitive psychology and constructivism, there has been a ‘social turn’ in mathematics education research since 1990’s. Before 1990’s, there had been 10 socio-cultural research reports in PME conferences (yearly 1 to 3 %), since then more than 10 % of yearly research reports have been socio-cultural (Lerman, 2006).

It should be noted that new theoretical frameworks have not completely overthrown the earlier frameworks. Instead they have typically increased the variety of theoretical frameworks used.

If one should make some predictions for the future trends in theoretical frameworks for mathematics education, embodied cognition holds a great promise. This view takes in to account the shared biology and fundamental bodily experiences of human beings as providing the foundation for mathematical thinking (Núñez, Edwards & Matos, 1999). This approach is likely to be informed by neurophysiologic research.

Trends in research topics

The following analysis on trends of PME research is based on the author selected research category for their research report. This category is picked from a list of categories identified by the PME International Committee. There have been some changes in categories over years, but it was still possible to identify the continuity of categories from one year to another, especially among the more popular categories. Moreover, some small categories were united with more popular ones. In order to balance the year-to-year variation of category popularity somewhat, I chose to combine 2-3 proceedings into one unit of analysis.

From PME categories, 18 most popular categories were identified and remaining categories were united with these. The popularity of each category will be presented in three graphs. In FIGURE 1 are the most popular topics. In this chart we can see one of the most notable changes over this period, namely rapid increase of reports on teacher education and professional development.

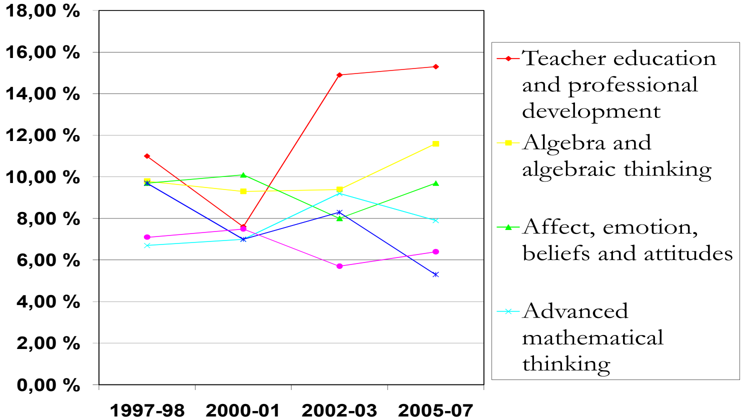

Figure1: The six most popular research topics in PME 1997-2007

In the next chart (FIGURE 2), we can see clear decline of the theories of learning and epistemology. This relates to the settling down of the debate over constructivism and its different interpretations. We can see a clear decline also in the topic "language and mathematics". The other changes in popularity are not equally clear, but we can say that socio-cultural studies and early mathematics have an expanding trend.

The third of the charts (FIGURE 3) presents the popularity of topics that cover less than 5% of the research reports each. Although these are clearly not trendy research topics in PME, we can see the re-emergence of proof, proving and argumentation as a relevant research topic.

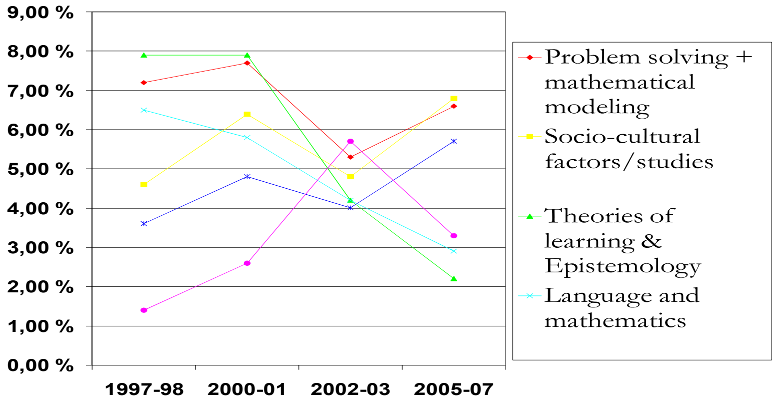

Figure 2: The six second most popular research topics in PME 1997-2007

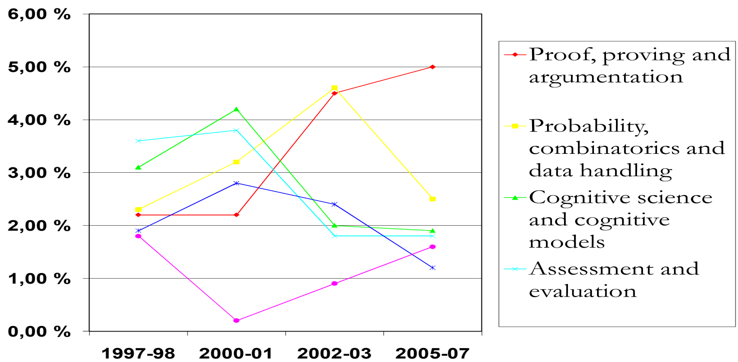

Figure 3: The six less popular research topics in PME 1997-2007

As a summary, we can see that algebra, affect, and advanced mathematical thinking have remained as popular topics, each covering 8-12% of yearly reports. We can also see the raise of teaching, teachers and teacher education as a very strong focus of PME research. There are two topics that have been popular earlier and that have regained their popularity after a period of inattention. Early mathematical development was popular topic in the early years of PME (Mulligan & Vergnaud, 2006). Then the focus was to understand learning counting and elementary arithmetic. More recent interest in these topics has been due to addressing a wider scope of contents as part of early numeracy. A similar developmental shift has taken place in research on proof and proving (Mariotti, 2006). Originally proof and proving were addressed as aspects of mathematical formalism and rigour, often relating to older students' learning of more advanced mathematical topics. More recently, proving and argumentation has been acknowledged as an essential part of mathematics knowledge building for all. This 'proof for all' view has made the topic more popular.

Somewhat surprisingly the computer tools and their influence on learning do not show any increase in Figure 1. On the contrary, there is a slight decline. However (Ferrara, Pratt & Robutti, 2006) point out that computers have become increasingly important aspects of mathematics teaching. Moreover, the focus has shifted from students' use of technology into design of tasks and teaching sequences. An explanation for this trend not to show in PME research reports is that computers have become mainstream. The computer is no longer the 'exotic' focus of the research, but simply a learning tool among others.

Methods of research

Emphasis in PME has over the last decade been in qualitative methods. For example, 66% of submitted proposals in 2009 were qualitative, 23% quantitative and 11% theoretical (Novotna, 2009).

Geography of Mathematics Education Research

I will also present a personal reflection on geography of research of mathematics education, inevitably biased due to own European perspective and activity within PME. Europe seems to be characterised by diversity. France has an emphasis on elaborate theoretical frameworks, such as didactical engineering and didactical contract. Their research has some influence worldwide, but French frameworks seem to remain isolated from other research. In The Netherlands, there is a strong tradition founded by Fishbein, Realistic Mathematics Education being their flagship. The Eastern Europe seems to be slowly turning to west, adopting western frameworks and research agendas. Russia and many eastern European countries have to a large extent remained outside the PME community. There the main interest seems to lie on education of the talented few, with a special interest on excelling in mathematics Olympiads.

Mathematics education research in the US has been strongly influenced by their local politics, most notably the ‘math war’ (Schoenfeld, 2004).

Canada seems to have a strong emphasis on qualitative methods, while Australia has perhaps the most balanced research program both in terms of research methods and in research topics. Israel has a strong mathematics education research community that has tight links to European, North American and Australian research.

Japan has its own tradition with lesson studies and problem solving and South Africa has special emphasis on equity and language.

Other established mathematics education communities exist in Brazil, Mexico, Taiwan and China. In case of China, an important factor is their collaboration with American Chinese researchers. Countries that have not been visible until recently, but that seem to be expanding in a rapid pace, are Turkey and (South) Korea. There are also a number of countries that have not yet been visible in these circles. Most notably many countries in Asia, Africa and South America are still only marginally involved.

Discussion

One thing that needs to be noted is the expansion of mathematics education research. Another clear change has been the diversification of research; the new approaches have not replaced the old ones, rather they have made the spectrum of research richer. Research seems to be also increasing in complexity both theoretically and methodologically. Mathematics education research uses more elaborate methods and combines often qualitative and quantitative methods. There seems to be also a trend of increasing connectivity, researches explicating how their research framework relates to other frameworks.

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MATHEMATICS EDUCATION RESEARCH AND RESEARCH EDUCATION IN THE NORDIC COUNTRIES

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Abstract

This paper has two parts. Part 1 presents examples from Nordic research in mathematics education and part 2 illustrates the research education in the Nordic countries.

International research in mathematics education

Mathematics education (ME) research started to expand in the 1960ies internationally. Scientific journals for reporting research studies in mathematics education were established (like Educational Studies in Mathematics in 1968). International congresses became more frequent (like the International Congress of Mathematical Education, ICME from 1969) and the community of researchers was growing. In 2008 the Centennial Jubilee of the International Commission on Mathematics Instruction (ICMI) was celebrated in Rome (Grevholm & Lepik, 2008).

Nordic research in mathematics education

In the Nordic countries a similar expansion came much later and the first professorships of mathematics education were created in the beginning of the 1990ies. For example Tapio Keranto in Finland, Anna Kristjansdottir in Iceland, Gunnar Gjone in Norway, and Mogens Niss in Denmark became professors in mathematics education. This did not lead to the construction of doctoral programmes in mathematics education in most of the places, but programmes in general education were in action and continued to be used. For at least 10 years there was most often only one professor in this field in each of the Nordic countries. Around 2000 the community started to grow and researcher education was established on more permanent basis. The scientific journal Nordic Studies in Mathematics Education, NOMAD, started in 1993. The first of a series of Nordic research conferences (NORMA) in mathematics education was initiated in 1994 by Erkki Pehkonen (Pehkonen, 1995). But still few new doctors were produced before 2000. The first professorship in mathematics education in Sweden was established in 2001 at Luleå University of Technology. A Graduate School in Mathematics, Physics, and Chemistry didactics started in the mid 1990ies in Finland. Nordic research in mathematics education does not have a specific signature or differ from international studies. Over the latest 10 years one may discern some themes that seem to be more common among Nordic researchers. As we will see below the international trends for research interests are often followed by the same trends in the Nordic countries.

First, I will exemplify by some themes found among new doctoral studies and then mention work done by groups of more experienced researchers. The following are examples of themes found: Studies on mathematics teachers and teaching, studies of mathematics textbooks and texts, studies of specific areas of the mathematics curriculum or development of mathematical concepts, studies of mathematics teaching and learning in the classroom, studies of use of ICT in mathematics teaching and learning, and studies of the history of mathematics related to learning and teaching. The method for finding the examples has been to go through all known theses in mathematics education in the Nordic countries since year 2000 and to use established personal contacts with doctoral students and supervisors in the Nordic Graduate School in Mathematics Education. But as space is limited for this paper I will not be able to mention all theses or active research groups.

Studies of mathematics teachers and their teaching

Kirsti Hemmi (2006) did a case study on the culture of proof in undergraduate mathematics courses at Swedish universities. As one part of her study, she inquired into the views of proofs by mathematics teachers at university level. She used a socio-cultural perspective on learning. The methodology she classifies as a picture drawing case study. It is primarily a descriptive account where she draws together the results of explorations and analyses of the phenomenon proof in the context of university mathematics in Sweden. She combines quantitative surveys with quantitative and qualitative document analysis when studying textbooks. Among the results she notes that mathematics students wonder what proof is and complain about the lack of discussion on the issue. They feel that it is implicitly expected, that they know what it is all about. The view of proof changes after the first oral exam.

Lisser Rye-Ejersbo (2007) studied the connection between an in-service course on teaching mathematics through open practical problems and the teachers' changed understanding and teaching following this course. She calls her study 'Design and redesign of an in-service course: the interplay of theory and practice in learning to teach mathematics with open problems'. She investigated how teaching, with open practical problems, is practised in different Danish classrooms. The course is part of a cycle of courses, and she labels the work design-based research. The design is used to develop the courses. The theoretical framework used, is drawn from theories of listening and responding, and from cognitive psychology dual process theory. The study led to the insight of how difficult it is to listen in new situations. As a consequence, the theoretical framework of virtual monologues was transposed into practice. From the study the teachers learnt how they were listening and it surprised them and helped them to take a step toward a reflective way to develop their teaching.

Bodil Kleve (2007) investigated how teachers in Norwegian lower secondary schools implemented the mathematics curriculum, L97, in her thesis 'Teachers implementation of a new curriculum in Norway, L97'. The methods used are focus group interviews, teachers' self-estimation, and classroom observations. She found different degrees of coherence between what the teachers say that they do and what they actually do in the classroom. Bodil Kleve used an ethnographic approach, and based on conversations in the focus groups, she selected four teachers to follow more deeply. The self-estimation draws on work by Pehkonen and Törner (2004) in terms of mathematics seen as a tool box (doing mathematics means working with figures, applying rules and procedures, and using formulas), from the system aspect (mathematics is a formal, rigorous system), and the process aspect. Her main source was teachers' reflections related to classroom observations. Each teacher estimated his own teaching, the ideal teaching and teaching according to L97.

Per Sigurd Hundeland (2007, 2009) collected data from three teachers in upper secondary school in order to investigate teachers' goals and what they emphasise in their teaching. He followed three mathematics teachers in their classes, observing them and listening to their reflections before and after lesson. He also interviewed teachers about their goal and aims, relating this to curriculum and the actual actions in lessons. He found that teachers are referring to their own teaching experience rather than their teacher education when they argue for their decisions and that a task–discourse exists also among these teachers in upper secondary school, as shown by Mellin-Olsen (1991) for compulsory school. Teachers use arguments about the frame factors that steer them to explain their decisions. Factors mentioned are lack of time, pressure from examinations, and pupils' lack of pre-knowledge from compulsory school.

Other authors in the area of teachers and teaching are: Claire Berg (2009) at University of Agder with the thesis 'Developing Algebraic Thinking in a Community of Inquiry', Tone Bulien (2008) at Tromsø University with the thesis 'Mathematical experiences in teacher education: a phenomenologically oriented analysis of students' texts' (in Norwegian), Ingvald Erfjord (2009) at University of Agder with 'Teachers' implementation and orchestration of Cabri use in mathematics teaching', and at Roskilde University Stine Timmermann Ottesen, whose work is relating teaching practices at university level with students' solution practices.

Studies of mathematics textbooks and texts

Monica Johansson in her licentiate thesis (2003), 'Textbooks in mathematics education: a study of textbooks as the potentially implemented curriculum' asked: What is the role of textbooks as a link between curriculum and activities in the classroom? To illustrate the textbook as the potentially implemented curriculum a content analysis of a textbooks series was conducted. The development of a commonly used textbooks series in Sweden is portrayed in the light of the curriculum development. Some findings from the analysis of textbooks show that the goals of mathematics teaching, as they are expressed in the national curriculum, are only partly realized.

In the second part of the study, 'Teaching mathematics with textbooks. A classroom and curricular perspective' (2006), she investigated how teachers use the textbook in the classroom. Here she chose to use data from the KULT-project in Uppsala. She worked with their video recordings from lessons and followed three teachers using different textbooks. Monica developed an instrument for analysis, which is based on the first part of her study and on earlier research. Her results show that students are only working with tasks in the textbook during the individual work of the lesson, which is more than half of the lesson time. In the teacher presentation the examples and tasks are mainly from the book. Mathematics as a scientific discipline is presented by the teachers in the same way as it is done in the textbooks. Teachers sometimes get into difficulties during lessons because they rely too much on the textbook.

Anna Brändström (2002) carried out an investigation of textbooks for year 7 in compulsory school in Sweden. She focused on the structure of the books and the building blocks in them. The outcome is that the different book series have great similarities. Later in her licentiate thesis (2005) she investigated differentiated tasks in grade 7 mathematics textbooks. She constructed an instrument of analysis with four aspects: pictures, operations, cognitive level and level of demand. Three commonly used textbook series were analysed. Results show that they are very similar. The authors do not use the opportunities to present differentiated tasks well. Astonishingly little use of functional pictures can be found, sometimes even less in low level tasks than in higher level tasks.

Other authors in the area of mathematics textbook studies are: Teresia Jakobsson-Åhl (2006) at Luleå University of Technology with the licentiate thesis *Algebra in upper secondary mathematics. A study of a selection of textbooks used in the years 1960-2000 in Sweden,* Niklas Bremler (2003) at Stockholm University with his licentiate thesis *Matteboken som redskap och aktör: En studie av hur derivata introduceras i svenska läroböcker 1967-2002* (The mathematics textbook as tool and actor: A study of how derivative is introduces in Swedish textbooks 1967-2002), Magnus Österholm (2006, 2008) at Linköping University with the thesis *Kognitiva och metakognitiva perspektiv på läsförståelse inom matematik* (Cognitive and meta-cognitive perspectives on understanding in reading in mathematics)*,* Kirsti Hemmi (2006) at Stockholm University with the doctoral thesis *Approaching proof in a community of mathematical practice,* Mira Randahl at University College of Narvik with an ongoing study of engineering students use of mathematics textbooks, and finally Tom Rune Kongelf at University of Agder also with an ongoing study of mathematics textbooks for compulsory school with focus on problem-solving strategies. Andreas Christiansen at University of Agder is studying mathematics textbooks in Norway from the beginning of 1900.

Studies on the development of mathematical concepts

Per Nilsson (2006) studied how pupils in grade 7 treat the concept of probability in an experimental situation, based on problems given in relation to games using sums of dice. A learning perspective was used, with the aim of describing students' ways of contextualising such probability problems. The data was analysed using an intentional analysis and provided a basis and meaning to the students actions. The results show the importance of relating students' conceptualising probability to their ways of creating meaning in a task situation.

Markus Hähkiöniemi (2007) investigated the concept of derivative. He studied the role of different symbolic and non-symbolic representations in problem-solving and in the learning of the derivative. Five students were chosen for task based interviews after a five-hour teaching period. There the derivative concept was introduced emphasising different representations and the open approach of the tasks. Based on the interviews a model of one possible learning path was constructed. The thesis was defended at University of Jyväskylä as was also the next one.

Antti Viholainen's study (2008) examines informal and formal understanding of the concepts of derivative and differentiability and the use of informal and formal reasoning in problem solving situations. The data were based on a written test given at six Finnish universities to 146 mathematics education students in the middle or in the final phase of their studies and on some oral interviews. An explanatory mixed method design was used. Connecting informal and formal reasoning was often difficult for the students. In particular, the students seemed to avoid using the definition of the derivative in problem solving situations. This created an obstacle in problem solving processes and in some cases led to wrong conclusions. Lack of ability to use the definition can not fully explain this tendency, as several students were able to use the definition when they were asked to do so. Viholainen recommends that the teaching of mathematics should support the development of coherence of students' knowledge structure and strengthen the understanding of connections between informal and formal representations.

Other studies on the development of mathematical concepts: Kristina Juter (2006) with the thesis *Limits of functions. University students' concept development,* Örjan Hansson (2006) with the thesis *Studying the views of preservice teachers on the concept of function,* and Kerstin Pettersson (2008) with the thesis *Algoritmiska, intuitiva och formella aspekter av matematiken i dynamiskt samspel ; en studie av hur studenter nyttjar sina begreppsuppfattningar inom matematisk analys* (Algorithmic, intuitive and formal aspects of mathematics in a dynamic interplay. A study of how students use their conceptions in mathematical analysis). Most of the studies on mathematical concepts were obviously undertaken at tertiary level.

Studies on history and didactics of mathematics

Some of the studies in this area have their focus more on history than on didactics and Ole Skovsmose (2006) in an evaluation classified those licentiate studies not to be inside mathematics education research. Thus we just mention shortly the four theses at Uppsala University by Kajsa Bråting (2009) *Studies in the conceptual development of mathematical analysis,* Johan Prytz (2007) *Speaking of geometry. A study of geometry textbooks and literature on geometry instruction for elementary levels in Sweden 1905-1962, with special focus on professional debates*, Sverker Lundin (2008) *The mathematics of schooling. A critical analysis of the prehistory, birth and development of Swedish mathematics education,* and Johanna Pejlare (2008) *On axioms and images in the history of mathematics.* Other studies are involving history in a different way more related to teaching, such as for example Uffe Jankvist (2009), who studied the use of history in mathematics teaching. A few more studies in history of mathematics are ongoing, such as the ones by Kristine Lohne and Andreas Christiansen at University of Agder.

Studies on use of technology in mathematics teaching and learning

Studies on the use of tools and artefacts in mathematics teaching and learning have been of importance internationally and they are also common in the Nordic context. Just to mention a few of them we list Lil Engström (2006) *Möjligheter till lärande i matematik* (Opportunities for learning in mathematics), Mette Andresen (2006) *Taking advantage of computer use for increased flexibility of mathematical conceptions,* Ingvald Erfjord (2009) *Teachers' implementation and orchestration of Cabri use in mathematics teaching,* and Mary Billington (2009) *Processes of instrumental genesis for teachers of mathematics. A case study of teacher practice with digital tools in an upper secondary school in Norway*. Mario Sanchez Aguilar is investigating changes in the theory-practice relationship when online education is used for in-service teachers. The theme of use of technology has also been in focus in Nordic conferences such as Norma08 with contributions from several of the mentioned doctoral students and new doctors (Grevholm, 2009). Some earlier examples of Nordic studies have been discussed in Grevholm (2006).

Nordic research in mathematics education other than doctoral studies

Doctoral study is actually part of an education and done during a time of apprenticeship for the student. Research done by experienced Nordic researchers should also be acknowledged. Several research groups are active in mathematics education, running larger projects and publishing papers and reports. These are not so easy to find as they can be in any of a number of national or international journals or in local report series. Research groups in ME at universities in Stockholm, Göteborg, Umeå, Linköping, Helsinki, Vasa, Oslo, Kristiansand and Roskilde must be mentioned but there are also single senior researchers who are active in different other places.

The MERGA-group (Mathematics Education Research in Agder) at University of Agder has run three large Norwegian Research Council-funded projects over three years each and reported extensively about them. The projects are developmental research studies where didacticians work together with mathematics teachers. The aim is to develop mathematics teaching in order to achieve better learning of mathematics (see Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild, & Grevholm, 2007).

At University of Gothenburg several researchers are active in didactics of mathematics. Ulla Runesson writes about use of theory of variation in the study of teachers' work, Per Olof Bentley's doctoral thesis was about Teachers' teaching and later he worked with data from the TIMSS-studies, and Johan Häggström works inside learning studies, and has written about pupils' work with linear equation systems. Thomas Lingefjärd studies use of ICT in mathematics teaching, Mikael Holmquist is studying student teachers' learning geometry, Lars Mouwitz are working in philosophy of mathematics. The mentioned researchers do belong to different groups in Gothenburg, one group in pedagogy, one in didactics of mathematics and one group working at NCM (National Centre for Mathematics Education).

At Roskilde University Mogens Niss, an active internationally recognised researcher, has written about mathematical modelling, assessment, and also overview papers of different aspects, such as the role of theory in ME, goals of ME, trends and characteristics of ME research, and his colleague Morten Blomhöj is interested in mathematical modelling, teachers and teaching, and use of ICT.

At University of Helsinki Erkki Pehkonen has studied attitudes and beliefs, teachers' work, student teachers' learning among other things, and Markku Hannula also investigated attitudes, beliefs and emotions, and gender issues. Marja-Leena Viljanen is working with digital tests in upper secondary mathematics, and Pavel Shmakov with humour in mathematics teaching.

At Umeå University Johan Lithner writes about mathematical reasoning, teachers, and student teachers, Peter Nyström worked with assessment and pedagogical measurement, Torulf Palm's area is authentic mathematical tasks, and Tomas Bergqvist studied calculators in mathematics learning, Jesper Boesen has studied assessment and national tests, and Eva Taflin's work is about the use of rich mathematical tasks. The group had several projects funded by the Swedish Research Council.

Space does not allow me to mention all that deserve to be mentioned. There is a need to find a way to get access to all the local research reports in mathematics education and an overview of published papers. A need to synthesise what we have learned from these reports becomes more evident as it is noticeable that new doctoral theses often neglect to mention relevant Nordic studies. NoGSME has created a database, which might for the future open opportunities to expose work from groups and individuals that deserve to be mentioned here.

What kind of studies do we not find among Nordic ones?

Not many studies on gender and mathematics education can be found. Exceptions are the theses by Riita Soro (2002) and Lovisa Sumpter (2009). Few studies concern socio-political issues related to mathematics education, exceptions are work by Paola Valero and Ole Skovsmose. Very few comparative studies exist and few studies with a longitudinal development in mathematics education. Exceptions are the studies on student teachers by Grevholm (2005) and Bjerneby Häll (2006). No studies on the history of mathematics education are visible. There are not really any curriculum studies. Further we find few studies with a critical mathematics education perspective (Skovsmose, 2006). Rather few studies concern mathematics teacher education, although this is considered a problematic area and many curriculum changes have taken place. Few studies try to synthesise what we already know from earlier studies, one recent exception is the paper by Persson (2009).

Doctoral education in Mathematics Education in the Nordic countries

The doctoral theses discussed above were produced in doctoral programmes in mathematics education or general education in the Nordic countries. Now let us take a closer look at these education programmes.

The doctoral programme in ME at University of Agder (UiA)

The programme started in 2002 (building on experiences from a master programme from 1994 with 80 master theses so far). Since the beginning 30 doctoral students have been taken up in the programme and in 2009 22 doctoral students are active in the programme, 6 have finished with a doctoral degree, and two theses are being evaluated. We expect another five to finish within a year. The students come from all over Norway and from abroad.

The faculty members are six professors, two docents, four lecturers, and one post doc in ME. Six doctoral courses are given regularly. UiA has probably the largest programme in the Nordic countries (in number of students, courses and faculty). The department is host for the Nordic Graduate School in Mathematics Education (NoGSME) during 2004-2009. University of Agder was created on the 1st of September 2007 from Agder University College. The core knowledge in the doctorate at UiA consists of a course in Theory of science (including ethical issues in research) and Methodology in mathematics education research. Participation in seminars in ME is recommended. The labour market is good, there is a need for teacher educators, school advisors and researchers in ME.

Doctoral programmes elsewhere in Norway

The programme at UiA is the only one in ME in Norway but there is a group active in natural science didactics (including ME) as part of a programme in general education at University of Oslo. General education programmes exist in Bergen, Tromsö and other places and offer opportunities for theses in ME. In addition professors in mathematics education are found in four other universities or university colleges. The doctoral programmes are of three years duration with one or two semesters of course work. A public defence of the thesis takes place and normally dissertations are published. So far in history less than 20 Ph Ds exist in ME in Norway.

Doctoral programmes in the other Nordic countries

Finland and Iceland do not have any special doctoral programmes in ME but students can take a mathematics education study in a general education programme. Half way degrees, licentiate degree, exist in Finland and Sweden. For more details see Grevholm (2007, 2008).

The structure in general is given in the table below:

Country	**Duration**	**Evaluation**	**Defence**	**Publication**
Denmark	3 years	Pre-examin.	Public	Trad. Differs
Finland	4 years	Pre-examin.	Public	Published thesis
Iceland	3 years	Pre-examin.	Public	Published thesis
Norway	3 years	Pre-examin.	Public	Published thesis
Sweden	4 years	No pre-exam.	Public	Published thesis

Cooperation in national Graduate Schools

Finland has a Graduate School in science and mathematics education since 1995 and Sweden had a Graduate School in ME 2000-2006 where 21 students were taken up. Denmark has a Graduate School in science and mathematics education with financing for 7 students.

Through collaboration between UiA and University of Oslo a Norwegian Graduate School in mathematics and science didactics is growing but it has not yet any external funding.

The Nordic Graduate School in Mathematics Education, NoGSME

The aim of the Nordic Graduate School is to support and develop the education of researchers in mathematics education in the Nordic and Baltic countries, to create constructive cooperation in order to raise the scientific quality of research in mathematics education, to give all doctoral students in mathematics education access to the activities of the Graduate School, and to create cooperation among a greater group of doctoral students and supervisors in order to share experiences and opportunities to improve the education of researchers.

The utmost aim is to create a network of cooperating partners, who can continue to collaborate after the five years of the Graduate School. Via NoGSME doctoral courses have been offered. Some of them are:

Theory of science from a mathematics education perspective, Methodology in mathematics education research, Meta-perspectives on mathematics and the learning of mathematics in a technological environment, History of mathematics with emphasis on modern mathematics, Problem solving in mathematics education, Theories of learning and teaching mathematics, Theoretical aspects of mathematics education with emphasis on the French School, Views of knowing and learning: Constructivism and socio-cultural theory, Gender and mathematics education, Justification of research in mathematics and science education with special emphasis on the role of theory in such justification, and Research on assessment in mathematics education. More than 120 doctoral students had access to these courses and 120 supervisors had access to supervisors' seminars where competence development was offered.

Some features to strengthen quality of doctoral education

There is always an ongoing debate about quality of doctoral education and theses (Grevholm, 2008). Ninety percent seminars have been used as one feature to assure quality and international studies are part of programmes in order to raise the level. Different models for supervision and competence development for supervisors have been used and are important especially for new comers among the supervisors. All students have at least two supervisors and there is a public defence of the dissertation to make it possible for anyone to discuss the scientific quality. External evaluation of the doctoral programme takes place regularly. Collaboration with international partners and use of international experts as opponents function as support for keeping quality issues alive.

Crucial or critical issues for ME doctoral programmes

Access to experienced supervisors in a rather new research field is a problem and thus competence development of newcomers is crucial. ME research is interdisciplinary and the inter-subject collaboration is not always smooth. Issues of format (monograph or collection of papers) and language (mother-tongue or English) in theses are debated and no easy answers exist. The financing question during and after the dissertation creates much work and there exists a vulnerability of small research environments. Opportunities to finance collaboration in graduate schools or Nordic networks are not easily found. The wish to finish within the expected time of the education can not always be met for different reasons. Many of these crucial questions are the same for all doctoral programmes and there is a need for a forum for critical debate about quality of doctoral theses. Such a forum does not yet exist.

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RESEARCH IN MATH EDUCATION IN LATVIA

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Abstract

The most important results and current activities in research in math education in Latvia are considered in the paper. The main accent is put on the three universities – the University of Latvia, Liepāja University and Daugavpils University, which carry out mathematics teachers' study programs

According to the structure of science accepted in Latvia, didactics of mathematics is considered as a part of mathematics, not as a part of pedagogics. Moreover, didactics of mathematics is included in one of 12 branches of mathematics, "Modern elementary mathematics and didactics of mathematics".

Mainly on the basis of the work of A.Andžāns (Andžāns, 1995), modern elementary mathematics was first in the world recognized as an official sub-branch of science in 1995 by the decision of Latvian Council of Science; since then, similar decisions have been made also in other countries. Following this approach, also didactics of physics, didactics of informatics etc. are included into corresponding branches of science in Latvia. This reflects the strong opinion that the inner logic of the scientific discipline has the dominant role in developing its didactics.

At this moment, the development of research in math education is carried out mainly in three universities: the University of Latvia, Daugavpils University and Liepāja University. Master and doctoral degrees can be obtained there; close cooperation has been taking place for decades. There are also individual researchers in other institutions.

Characteristic features of research in all abovementioned universities are close ties between research and educational praxis. In many cases the promotion work results are based on the experience accumulated by running some study program, using a series of textbooks or teaching aids, etc.

The main directions of research in mathematics education are:

1) modern elementary mathematics,
2) possibilities and problems in advanced teaching of mathematics to middle and high school students,
3) research in general math education,
4) research in higher education,
5) possibilities and impact of ICT on teaching mathematics,
6) research in the history of mathematics.

Modern elementary mathematics and advanced teaching of mathematics

The term "elementary mathematics" often is considered as "school mathematics", nevertheless the meaning of it is much broader. In (Andžāns, 1995) elementary mathematics is described as follows.

> "*Elementary mathematics consists of:*
> *1) the methods of reasoning recognized by a broad mathematical society as natural, not depending on any specific branch of mathematics and widely used in different parts of it,*
> *2) the problems that can be solved by means of such methods.*"

Evidently, such a concept of elementary mathematics is historically conditioned.

Many new areas of mathematics, especially in the discrete and algorithmic parts of it, are still today exploring elementary methods as the main tool.

Modern elementary mathematics has also close connections with advanced math education at school, even more – the movement of mathematical contests has made an important service to elementary mathematics. The system of math competitions created a large and constant demand for original problems on various levels of difficulty. However school curricula couldn't settle the situation, and the organizers of the competitions turned to their own research fields where they found rich and still unexhausted possibilities.

One of important results that originated from the "olympiad mathematics" was the identification of the so-called general combinatorial methods:

- mean value method,
- invariant method,
- extremal element method,
- interpretation method.

The research in the area of modern elementary mathematics and advanced math education mainly is concentrated in the University of Latvia, particularly in the scientific laboratory A.Liepa's Correspondence mathematics school (CMS). CMS is founded in 1969 and has developed new approach in mathematics education that is based on math competitions and even partial analysis of high-level problems.

Main research results and topics in these areas are:

- PhD thesis of Līga Ramāna about the role of the method of invariants in math compactions and the possibility to introduce method of invariants in school's mathematics curricula (Ramāna, 2004)
- PhD thesis of Dace Bonka "The interpretation method in elementary mathematics and mathematical competitions for the elementary school age students" (2008) about usage of

interpretation method suitable for school students and system of math competitions for basic school students (Bonka, 2008).

- Ingrīda Veilande does her research about usage of mean value method in advanced math education.
- Aija Cunska does her research about the role of mathematical induction in math education.
- Andrejs Cibulis does his research in the area of polyominoes (Cibulis, 2001) and other mathematical puzzles and elementary methods for solving extremum problems (Cibulis, 2003-2006).

Research results in these areas have gained high international reputation.

General math education in school

Research in the general math education is mainly carried out at the University of Latvia, Liepāja University and the Daugavpils University.

Main research directions in this area are:

- new didactical approaches in teaching mathematics,
- new textbooks.

Prof. Jānis Mencis (sen.) (Liepāja University) is an undoubted and internationally recognized leader in the math education for junior students. He is author of more than 300 textbooks along with teaching aids and methodical literature for teachers.

An amount of information to be acquired at school increases significantly during some last decades, therefore also main approach in education overall and in math education particularly is changed – an accent is a shift from acquiring a large amount of facts to the training of practical skills how to work with information and use it. As geometry is the largest and, in fact, the only part of school mathematics curricula which is developed in a deductive manner as well as geometry is an appropriate language for teaching natural sciences, teaching of geometry is significant topic for math education researcher, e.g.,

- New concepts in teaching geometry for secondary school students are developed by math teacher Elīna Falkenšteine. They are applied in textbooks series (Andžāns a.o., 1992–1998).
- PhD thesis of Ilze France "Teaching of geometry in the basic school (Grade 7-9)" (defended in 2005).

Another trend of studies is the role of combinatorics and combinatorial methods in the basic school. Gunta Lāce is working in this area.

Ilze France, Gunta Lāce at al. have developed new textbooks in mathematics for basic school, based on the new curricula along with methodical references for teachers in a "Teacher's book" (France a.o., 2007-2009)

Math education at school doubtless is also closely connected with math teachers training. Math teachers' higher professional and master study programs are offered at the University of Latvia (program directors Jānis Mencis (jun.) and Agnis Andžāns), Liepāja University (program director Edvīns Ģingulis) and Daugavpils University (program director Elfrīda Krastiņa).

ICT in teaching mathematics and math teaching in higher education

New research direction that becomes actual since last decades of previous century is the possibilities of usage of ICT in education. There was developed a project LIIS (*Latvian Education Information System*) (1997-2005) in Latvia. A lot of digital and interactive teaching aids in various subjects were elaborated within it.

Research in the area of math teaching in higher education and usage of ICT (both in school and higher education) is carried out in many institutions (e.g., University of Latvia: Agnis Andžāns, Inese Bula, Jānis Buls, Aija Cunska, Halina Lapiņa, Līga Ramāna etc., Liepāja University: Dzidra Krūče, Aija Kukuka, Dzintars Tomsons etc., Latvian Agricultural University: Sarmīte Čerņajeva, Ilze Jēgere, Anna Vintere etc.).

History of mathematics

Research in the history of mathematics in Latvia is closely tied with research in didactics, as it is included into mathematics teachers' professional study programs. So some methodical teaching aids for students in history of mathematics and history of didactics of mathematics in Latvia are elaborated, e.g., by Daina Taimiņa (Taimiņa, 1990), Edvīns Ģingulis (Liepāja University) (Ģingulis, 2009) etc.

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MATHEMATICS EDUCATION RESEARCH AND RESEARCH EDUCATION IN LITHUANIA

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Abstract

Areas where dissertations in mathematics education can be written and defended, short survey of articles in Lietuvos matematikos rinkinys (Lithuanian Mathematical Journal), a few areas of research and researchers are presented in this article.

Introduction

In this article very personal point of view about problems of research in the area of mathematics education will be presented. Author apologizes in advance researchers who perhaps will not be mentioned or mentioned not at sufficient deepness in this paper. The reference list is quite short, very often (almost always) only one article represents a researcher. Main task was not to register all researches and all articles but to to emphasize problems of writing and defending dissertations in mathematics education.

Research in mathematics education is very specific. It requires universal knowledge in many areas: mathematics itself, psychology, philosophy, sociology, pedagogy, statistics and others. This universality causes a problem – in what area to defend dissertations. At the beginning we shall discuss academic environment in Lithuania where scientific activities in mathematics education are cultivated. Later on we concentrate on publications in special issue of Lietuvos Matematikos Rinkinys (Lithuanian Mathematical Journal) where articles of the Annual Conferences of Lithuanian Mathematical Society are published. The list of dissertations in mathematics education since the year 1994 is presented (see Appendix).

Doctoral studies

As was mentioned in the introduction, for a person who wants to do research and defend a dissertation in mathematics education the first problem arises of choosing the proper scientific area. According to the nomenclature of science in Lithuania there are such possibilities:

Physical sciences

Mathematics – 01P,

Informatics – 09P,

Social sciences

Education – 07S,

Humanitarian sciences

History – 05H.

We can see that there is no special area for mathematics education. Each of the areas, mentioned above, has its own rules, traditions etc. and very specific research in mathematics education is often not welcomed.

Let us list universities where doctoral studies are allowed in interesting us fields.

Mathematics – 01P

Vilnius University, Faculty of Mathematics and Informatics,

Institute of Mathematics and Informatics of Vilnius Gedimino Technical University.

Informatics – 09P

Vilnius University, Faculty of Mathematics and Informatics,

Institute of Mathematics and Informatics of Vytautas Magnus University (Kaunas).

Education – 07S

Vilnius University, Faculty of Philosophy, Department of Education, Kaunas University of Technology, Faculty of Social Sciences, Institute of Educational Studies,

Vilnius Pedagogical University,

Šiauliai Pedagogical University,

Vytautas Magnus University (Kaunas).

It sounds strange but perhaps the strongest centre in education is at Kaunas University of Technology under the guidance of prof. Palmira Jucevičienė.

The next problem is the supervisors for doctoral students. According to the law ISAK-625, issued 2006-03-31 by the Lithuanian Ministry of Science and Education supervisors for doctoral students must have published at least a monograph and one ISI publication or 3 ISI publications during the last five years period (at application moment). ISI publication is defined as the publication in the magazine listed by the Institute for Scientific Information. As we shall see later on, most publications in Mathematics Education area of Lithuanian researches are in proceedings of conferences, which do not belong to that list.

Conferences and publications

There are three regular conferences where Lithuanian mathematics educators traditionally take part.

1. The annual conference of Lithuanian Mathematical Society. The 50th conference took place in the year 2009. This conference has quite stable format. There are two sections: Didactics of teaching mathematics and informatics and History of mathematics.
2. Baltic conferences Teaching Mathematics: Retrospective and Perspectives.
3. Annual conferences organized by Kaunas University of Technology “Mathematics and teaching of mathematics”.

Conference Year	Number of articles	Lithuanian	English	Russian
2000	10	9	1	0
2001	11	10	1	0
2002	20	18	1	1
2003	22	20	1	1
2004	22	20	1	1
2005	18	17	0	1
2006	21	21	0	0
2007	23	23	0	0
2008	17	14	1	2

Table 1: Language distribution of articles

There are also many other international or local conferences where Lithuanian researchers participate.

In Table 1 we present the numbers of articles published in special issue of Lietuvos Matematikos Rinkinys during the period 2000 – 2008. Almost all papers are in Lithuanian, except few in English and in Russian. In table 2 we present a bit statistics about main topics of papers published in the period 2000–2008. Numbers are approximate because sometimes (or often) a paper can be ascribed to different topics.

Abbreviations in Table2: Primary – primary education, IT – application of IT in mathematics education, Un – university education, HS Cur – high school curriculum, Olymp – mathematical Olympics and contests, Ass – assessment, HS&Un – relations between high school and university, Didact – general problems of didactics, Term – terminology, Teachers – teacher education.

The list of dissertations defended since 1990 in mathematics education is presented in Appendix. Probably some mathematics education related dissertations from the field of social sciences are missing. One should emphasize the names of B.Balčytis and A.Ažubalis who have defended the second dissertations- doctor habilitatus. Since the year 2003 the second dissertation as such was cancelled and procedure of habilitation, necessary for professorship, was established instead.

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
History	3	2	3	6	6	5	4	7	4
Primary	2	1	1	2	1				
IT	1		1		2		1	2	1
Un	2		3		2	3	3		1
HS Cur	2	2	1				2	1	1
Olymp		1	2	1	1		3	2	1
Ass		1		2	3	1	2	2	2
HS&Un			2		1	3			
Didact			3	2	1		1	5	4
Terms			1	2	2				
Teachers							1		

Table 2: Topics distribution of articles

Research

We shall present here a few research areas and a few personalities, playing important role in Lithuanian mathematics education.

History. This part is sufficiently large (see table 2). Since 2005 at the annual conferences of Lithuanian Mathematical Society the section of Mathematics history was separated from Mathematical didactics. At first it should be mentioned two monographs of A.Ažubalis (1997, 2005) where mathematical didactics in Lithuania is considered from the XIX century till the year 1940 and from 1940 till 1990.

Very often biographies and activities of known mathematicians, authors of mathematical textbooks, and professors of mathematics of Vilnius University are presented. For example, prof. Zigmas Žemaitis (Kubilius, 2005), professor of Vilnius university in XIX century Zacharij Nemcevski (Gečiauskas, 2006), the honoured teacher Petras Rumšas (Jasiūnas & Verikaitė, 2007), author of many textboos for secondary school and didactician Pranas Mašiotas (Ažubalis, 2008). Also old mathematics textbooks were studied (Banionis, 2006).

Primary school. Mainly researchers from Šiauliai University Kiseliova & Kiseliovas (2003) are doing research on primary school mathematics. They also write mathematics textbooks for primary school. They continue work of professor and mathematics textbooks' author for primary school Bronius Balčytis, who also was doctor habilitatus (App. Balčytis, 1994).

Mathematical Olympiads and contests. Romualdas Kašuba and Juozas Mačys are very active in this area. They are involved in national olympiads, the International Mathematical Olympiads, team mathematical olympiad „Baltic way" , Kangaroo and many other contests. Kašuba regularly participates in German Mathematical conferences (Tagung für Didactic Mathematic) since 2001. He examines mathematical problems from psychological and estethical point of view (Kašuba, 2001). Mačys analyses contests problems (Mačys 2006). Prof. Valentina Dagienė is enormous active in informatics Olympiads. She studies informatics problems' relations with mathematics (Dagienė, 2004).

Terminology. Few articles concerning terminology (Pekarskas& Pekarskienė, 2003) are published.

Information technologies. The first to mention in this area is professor of Vilnius Pedagogigal University Joana Lipeikienė. She investigates applications of different software for teaching mathematics (Lipeikienė, 2000, Lipeikienė&Lipeika, 2006).

Semiotics. This topic is not popular in Lithuanian mathematics education research. In western research mainly Peirce approach of semiotics is used. Lithuanians have well known semiotician A.J.Greimas, who was the maître of French semiotics school. Kudžma (2005) tried to apply Greimas's approach of semiotics to study some mathematical texts, especially concerning inverse function.

Lithuanian school for young mathematicians is functioning since year 2000. A.Apynis, E.Stankus (Vilnius University) and J.Šinkūnas (Vilnius Pedagogical University) are conducting this school. Apynis, Stankus and Šinkūnas (2004) present information about first five years and prehistory of this school. Also these authors write about curiculum problems of high school (Apynis & Šinkūnas, 2007), (Stankus, 2007).

International studies. Lithuania took part in TIMSS (Trends in International Mathematics and Science Study) on 1995, 1999, 2003, 2007 and PISA (Programme for Interantional Student Assesment) studies on 2000, 2003, 2006, 2009. The initiator of TIMMS in Lithuania was A.Zabulionis. He published the first publication related with TIMSS (Zabulionis, 1997). Also he founded National Examination Centre (NEC). Participation in these interantional studies let to receive real objective data about many aspects of Lithuanian secondary school education in mathematics and sciences. These international projects are conducted by NEC, the current coordinator of these projects is A.Elihio. In 2006 she defended doctor dissertation in mathematics (App. Elihio, 2006), more precisely, in statistics. For applications she used in her dissertation the data from TIMSS.

Another worker from NEC J.Dudaitė defended dissertation in Education (App. Dudaitė, 2008). This dissertation is also based on data from TIMSS. It is important to add that NEC is becoming serious research center in mathematics education. But the data from TIMSS and PISA are still waiting for further analysis and research.

As we discussed at the beginning of this article, there are two different ways of writing and defending dissertations: via mathematics or via educology. From the point of view of the author of this article – the best way would be to have the doctoral program in Mathematics education. Let us hope that this will be realized in the future.

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APPENDIX: Lithuanian dissertations in mathematics education

Social Sciences, Education (07S)

Cibulskaitė, N. (2000). Matematikos mokymo humanizavimas V-je pagrindinės mokyklos klasėje [Humanization of mathematics teaching at the 5th grade of secondary school]. Vilnius Pedagogical University.

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Dabrišienė, V. (2001). Pedagogo veikla individualizuojant ugdymo programas Lietuvos bendrojo lavinimo vidurinėje mokykloje [Teacher's activity in individualization educational programs at Lithuanian secondary schools]. Kaunas University of Technology.

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Sičiūnienė, V. (2003). Statistikos ir tikimybių teorijos pradmenų mokymo Lietuvos pagrindinėje mokykloje sistema [The system of teaching elements of statistics and probability theory at Lithuanian secondary school]. Vilnius Pedagogical University.

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Dudaitė, J. (2009). Mokinių matematinio raštingumo kaita edukacinės ir mokymosi aplinkų aspektu [Changing of students' mathematical literacy with respect of educational and learning environment]. Kaunas University of Technology.

Humanitarian Sciences, History (05H)

Banionis, J. (1995). Matematikos mokslo raida Lietuvoje 1920-1940 m. [Development of mathematical science in Lithuania 1920 – 1940]. Vytautas Magnum University.

Physical sciences, Mathematics (01P)

Elijio, A. (2006). Imties klasterizacijos efektai statistiniuose švietimo tyrimuose [Some effects of cluster-sample design in statistical educational surveys]. Vilnius University.

Social Sciences, Education (07S), doctor habilitatus dissertations

Balčytis. B. (1994). Matematikos mokymo turinio ir metodikos optimizavimas lietuviškų mokyklų pradinėse klasėse 1970-1973 m. [Optimization of mathematics teaching content and methodology at Lithuanian primary school 1970-1973].

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MATHEMATICS EDUCATION RESEARCH IN ESTONIA

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Abstract

The paper describes some key characteristics of Estonian research and developmental work in mathematics education. After a short look at history, main research areas and methods used will be reviewed, also research communities and researchers' training system will be described.

Introduction

During the last four decades or so mathematics education has become established as an academic discipline on the international scene. To show this we need only refer to a number of facts, such as the existence of a multitude of departments in universities and research institutions; research grants and projects; academic programmes and degrees; international scientific organisations and bodies; journals and publication series, hosts of conferences; and so forth, all devoted to mathematics education (Niss, 1999). The situation in the academic field of mathematics education is relatively well analysed and documented in a number of international reviews (Grouws, 1992; Biehler et al, 1994; Bishop at al, 1996; Sierpinska & Kilpatrick, 1998; English, 2002).

The aim of this overview is to characterise how these international developments have been reflected on the national level, and the state of the art of Estonian mathematics education research is presented. The following analysis is largely based on my previous general knowledge of Estonian mathematics education research. Also the small questionnaire was directed to individuals in the departments. This questionnaire probed the substance of research – themes and methods and colleagues' views of the situation regarding research in mathematics education in the country.

The paper describes some key characteristics of Estonian research and developmental work in mathematics education. After a short look at history, main research areas and methods used will be reviewed, also research communities and researchers' training system will be described.

While assessing Estonian developments and results of mathematics education research one should consider the very small size of the country. It has clearly set the limits for research environments and resources available to carry out scholarly activities in the field. For example, there have never been permanent full positions for research in the field of didactics of mathematics and there are very few positions that are principally intended for advanced research at the post-doctoral level. Typically, academics should carry out several activities in parallel: teaching, curriculum development, textbook writing, developmental studies and research. Thus, most colleagues report doing research on a less than half-time basis. So, great amount of research has been performed by academic persons whose main occupation really is not research. This is why the research done is typically strongly connected to developmental work. And of course, such a situation could be counted as a strengh and as a weakness at the same time. Part-time involvement in research combined with part-time teacher education or part-time textbook writing is a strong combination. No doubt, having researchers involved in that work is beneficial. From the other side, serious post-doctoral level research requires concentration and full-time contribution which has not been possible in many cases.

History

Mathematics education in Estonia has a long history. Let me just mention some most important milestones to set the scene.

The first mathematics textbooks in Estonian language were issued already on the first half of the 19th century, the first one by Peter Hinrik Frey in 1806. By the end of the century all fields of elementary mathematics (arithmetic, algebra, and geometry) were well covered with original textbooks (Prinits, 1992).

In 1828 the regular training of teachers in Tartu Teachers' Seminarium was started and training to teach elementary mathematics was also included into the programme there. It is interesting to note that relatively soon; in 1874 the first methods' book for mathematics teachers was issued. There exists also the German language version of it (Kallas, 1899). The book is a very detailed description of what lessons in elementary mathematics topics should look like. Its author Rudolf Gottfried Kallas is giving concrete suggestions how to teach arithmetic, stressing the importance of visualisation and meaningful learning and criticising rote learning.

The proclamation of independence in 1918 gave also a start to active discussions and developments in education, incl. school mathematics. The changes in Estonian mathematics education at that period were well rooted into the international reform movement headed by the International Commission on the Teaching of Mathematics founded in1908 in Rome (Grevholm, Lepik, 2009). The new national curriculum was developed for lower and upper secondary mathematics, elements of functional-dependence, derivatives and integration were introduced. In concern to primary mathematics, there was a strong movement to make mathematics learning more active and more meaningful by introducing workbooks and using practical tasks (Prinits, 1992). In general, it was the period of active theoretical discussions about school mathematics and its teaching, lots of new study materials and textbooks were developed.

From the1940s the fifty years long Soviet period started and it has left its strong footsteps into Estonian mathematics education as well. This period is characterised by attempts to unify the content of school mathematics and its teaching methods all over the huge union. Estonia was the only republic having somehow the privilege to continue following its own curriculum and issuing its own textbooks. Thanks to that the continuation of traditions and the national school of thought in the field of mathematics education survived.

On the middle of 60s mathematics education is starting to position itself as academic discipline in Estonia. The first pieces of systematic research started to appear, the first PhD dissertation has been defended by Olaf Prinits and the first university- department of mathematics education was founded in Tartu.

In the 1960s and 1970s New- Math- movement emerged internationally and those ideas initiated active changes in Estonian school mathematics as well. The guiding principle in the New Math was the careful construction of mathematics concepts, beginning with sets and logic. So, problems connected with developing the new content of school mathematics and its teaching methods become the central interest of didacticians and were clearly reflected in the mathematics education research done.

This period proved to be extremely fruitful in mathematics education for Estonia. Lots of research projects were initiated and the remarkable number of PhD dissertations was defended (see Table1).

The shift in 80s brought the learner into the centre of researchers' interest. Student's mathematical abilities, ability grouping, learning processes and difficulties were studied also in Estonia.

In these years Baltic cooperation on mathematics education research and developmental work was initiated. The first joint seminar took place in 1984 in Liepaja, Latvia. In 1985 similar seminar was held in Tartu, Estonia, and the third one was arranged in 1986 in Vilnius, Lithuania. With those three, joint seminars started to take place on a regular basis. Since 1998 the seminars turned into the annual conferences *Teaching Mathematics: Retrospective and Perspectives*. Also the proceedings of the conferences started to appear (Azubalis & Gingulis, 2005).

Author	Year of defence	Topic
Olaf Prinits	1959	Methods of introduction of elements of higher mathematics into school mathematics
Aksel Telgmaa	1966	Methods of curriculum design in school mathematics
Jaan Reimand	1969	Mathods of teaching linear planning in secondary school
August Undusk	1971	Development of concept of a function in school mathematics
Kalle Velsker	1972	Methods of teaching probability theory and statistics in secondary school
Evi Mitt	1973	Methods of teaching set theory and mathematical logics in secondary school
Jüri Afanasjev	1974	Students' knowledge levels in lower secondary learning of mathematics
Reet Ruga	1975	Early teaching of mathematical concepts
Lea Lepmann	1982	Mathematics component in pre-service training of matematics teachers
Helle Sikka	1987	Differentiated learning of mathematics in primary grades
Tiit Lepmann	1987	Mathematical concept development
Madis Lepik	1989	The process of verbal problem solving: differences in solving difficulties

Table 1: Estonian dissertations in mathematics education

In the Soviet period research paradigms were strongly influenced by Russian educational ideology. Under the Soviet regime the interaction with the international community of researchers in mathematics education was nonexistent, even the Western research literature wasn't available. Only political changes in the beginning of 90s opened the world also for researchers and brought new ideas and developments into Estonian mathematics education. Scientific contacts were established with the rest of the world, mainly with colleagues from Central Europe and Nordic countries, participation in international researchers' community became a norm.

Research problems and methods

The range of problems that are studied in the field of mathematics education research can be listed along several dimensions. In the following I am going to review the main lines of research done in Estonia in chronological order.

In the 1960s and 1970s the mainstream research carried elements of didactical design and was strongly content oriented. The curriculum developers were inspired of the ideas of New Math movement and aimed at introducing new concepts into the school mathematics. The researchers followed the same trend. A number of projects were carried out to investigate the possibilities of introducing certain new topic areas (like probability theory, statistics, mathematical logics, set theory) into school mathematics and to design relevant teaching methods. Mathematics teaching methodology („matematikmetodik")- research that concerns the „how to do it in typical large-size classroom" component of mathematics didactics was dominant. Research methods used were empirical and quantitative, strict data analysis and criteria such as representativity, validity, reliability were in the centre of attention. Experimental designs, with experimental and control groups, using pre- and post-test techniques, have been used in many of these studies.

If one considers the PhD thesis in mathematics education defended at that time (by Prinits, Telgmaa, Reimand, Undusk, Velsker, Mitt, Ruga; see Table1), one notices that the content of school mathematics together with methods of its presentation were of central importance.

The shift in the 1980s, after the failure of the New Math, brought the learner into the centre of researchers' interest. It was recognized that deeper understanding of the learning processes was indispensable. Mathematics learning became an important research field also in Estonian mathematics education. Student's mathematical abilities, ability grouping, knowledge acquisition and learning difficulties were studied (thesis by Afanasjev, Sikka, Lepik; see Table1).

There was a strong line of research to study mechanisms of learning mathematical concepts. Extensive analysis of the status of mathematical conceptions as abstract objects and their methodical representations in textbooks, as well as the conception formation processes led to important insights. Using findings from cognitive psychology, model for the cognitive processes of formation of mathematical conceptions was constructed and the methods for teaching of concepts were developed (thesis by T.Lepmann).

Another PhD- project focussed on problem-solving processes and tried to investigate and explain students' difficulties in solving verbal problems, also the differences in solving processes of students with different mathematical abilities were studied (thesis by Lepik).

Research methods used in the studies described above were still mainly quantitative, based on statistical analysis of empirical data. In some cases also interviews as well as observations of the groups of learners or even an individual learner's mathematics construction processes were used.

Another well developed area of studies on the 1980s was connected to the history of mathematics education in Estonia. It was initiated by prof O.Prinits and followed by A.Undusk, L.Lepmann and others. The developments of the content of school mathematics and mathematics textbooks throughout two centuries were carefully studied and reported in numerous of publications. As the result of the decade- long studies three monographic volumes on the history of Estonian school mathematics were composed and published (Prinits 1992, 1993, 1994).

A further line of research (starting from the 1990s, still active) was the turn towards affective factors of mathematical learning. Inspired by the work done in Finland by prof. Erkki Pehkonen and others, students' and teachers' beliefs and attitudes raised into the focus of studies. There has been an emphasis on research focussing on understanding the students' beliefs about mathematics and its learning (L.Lepmann, Afanasjev). Also several larger scale comparative studies on students' mathematical beliefs have been carried out (L.Lepmann, Kislenko).

Teachers' and teacher students' beliefs about mathematics, its teaching and learning have been in focus as well (L.Lepmann, Lepik). This topic has been developed further during recent years by the group of researchers in Tallinn led by visiting professor Markku Hannula. Teachers' change and professional development in the framework of community of practice has been in the centre of interest of that research group (Hannula, Lepik, Kaljas).

As typical to belief-studies, mainly surveys on the basis of Likert-scale questionnaires followed by statistical data analysis have been widely used in those studies.

In addition to research described above a number of national survey studies ordered by National School Board have been regularly conducted during last decades. Those studies have assessed pupils' achievement and mathematical knowledge in grades 3, 6 and 9 and provide long term descriptions of mathematical skills of Estonian school children (T.Lepmann, L.Lepmann, Afanasjev, Jukk). It should be mentioned also, that in 2000 Estonia participated in TIMSS and in 2008 in PISA. The Estonian data from those international comparative surveys have been analysed also nationally (T.Lepmann, Lepik).

Research communities

I will now add some comments regarding the institutionalization of research in mathematics didactics in Estonia.

In many countries research in mathematics education was started and for a long period developed at departments of education or teacher training. Thus the people involved were educationalists with stronger background in educational psychology and often limited training in mathematics. Such setting definitely had its influence on the focus of research done. And only lately the mathematics education research has started to move to the departments of mathematics. For example in Sweden the first PhD thesis in mathematics education was defended in mathematics department only in 1996 in University of Lulea (Bergsten, 2002).

In Estonia the developments have been different. Here the research and teaching in mathematics education started from the very beginning at the departments of mathematics. So, in 1965 the department of mathematics education was formed in the faculty of mathematics in the University of Tartu and at the same time- period the section of mathematics education was initiated at the department of mathematics in Tallinn Pedagogical Institute (currently Tallinn University).

Those two centres are still the main institutional bodies carrying out mathematics education research and developmental studies in the country. The number of faculty members focusing on mathematics education in Tartu has varied from 4 to 7 and in Tallinn from 2 to 4. There is limited number of persons working outside these centres as well. Scholars in both of these institutions have traditionally had a background in mathematics and secondary mathematics teaching. Teaching obligations of those faculty members (in addition to research tasks) have always included also courses in mathematics. And this is something what makes it complicated for persons without a background in mathematics (for example from education or psychology) to enter the field in a natural way. Such a situation explains also why Estonian research in mathematics education has been dominantly content and teaching methods oriented.

Both these centres have presently unfilled professorships in mathematics education. Late professor Olaf Prinits who's main research interests were in the history of mathematics education worked in Tartu and professor emeritus Aksel Telgmaa with the main interests in the content of school mathematics in Tallinn. For the last three years Markku Hannula (Finland) served as visiting professor in Tallinn University. His research interests lay in the affective side of mathematical learning.

As far as the number of persons involved in mathematics education is small, they work in close cooperation. There exists also quite young organisation outside universities that unites researchers and mathematics education developers. Under the framework of Estonian Mathematical Society (EMS) the Association for Research in Mathematics Education was formed in 2008. The goal of the Association (as the subdivision of EMS) is to support research in mathematics education and dissemination of its results, to facilitate collaboration among researchers and developers of the field and international cooperation. Association has also applied for the membership in Nordic Society for Research in Mathematics Education, NoRME.

Doctoral education

Well functioning research education is one of the central aspects for every academic field. It is a big challenge for Estonian mathematics didactics as well. Due to the very small size of the field, it has not been possible to open an independent doctoral programme neither in the University of Tartu nor in Tallinn University. At the same time there has been the clear need for researcher education, assuring the continuity of the field. The problem has so far been solved in collaboration with the departments of education. In both universities there exists a module in didactics of mathematics as one possible option in the PhD programme of general education and it is possible to defend thesis in mathematics education. There is another option as well: the state provides full grants to take the degree abroad. The agreement between our universities and Nordic Graduate School in Mathematics Education has made this option especially operational (see also Grevholm, 2008).

Last years have demonstrated the increased interest among young researchers toward doctoral studies and so, both options mentioned above have been actively used.

At the University of Tartu Anu Palu and Hannes Jukk are currently taking their doctoral education. Anu Palu is finalising her thesis on learning primary mathematics, she is studying the development of pupils’ mathematical knowledge and skills. The doctoral- project involves also studies of primary school teachers’ beliefs about teaching mathematics.

Regina Reinup and Indrek Kaldo from Tallinn University are in early phase of their doctoral studies. Regina Reinup is studying affective factors of mathematics learning and searching for different motivational teaching methods. Her study is a developmental research with strong emphasise on the use of qualitative research methods.

Indrek Kaldo’s research area is university mathematics and its teaching and learning. He tries to investigate the reasons for the high drop-out rate in undergraduate mathematics courses. His research project involves also studying of students’ and faculty-members’ attitudes towards mathematics teaching and learning.

Lately two persons from Tallinn have also taken the chance to have their doctoral education from abroad. Kirsti Kislenko is finalising her doctoral studies at the University of Agder, Norway and Jüri Kurvits is studying at the University of Helsinki, Finland.

Kirsti Kislenko’s research interest is pupils’ affective domain in mathematics education, including beliefs, attitudes, and emotions in mathematics learning. In her doctoral thesis Kirsti is investigating Norwegian and Estonian 7th, 9th, and 11th grade pupils’ opinions about mathematics learning. Her main research instrument is a Likert scale questionnaire but she also uses qualitative research methods, like pupils’ and teachers’ interviews, and lesson observations.

Jüri Kurvits focuses in his doctoral research on pupils’ relational understanding and its development in school mathematics. During middle grades’ school mathematics pupils have to move from additive reasoning that characterizes the whole numbers, to quite different multiplicative reasoning. Qualitative change has to happen in pupils’ mathematical thinking and Jüri Kurvits is aiming to study these processes.

With a number of young people now preparing themselves in a systematic way, it is to be hoped that there actually will be careers available for them inside Estonian mathematics education, when they reach their doctor’s degree.

Summary

Mathematics education research is still very small area of scholarly activities in Estonia.

Today, measured in number of publications and high level conference presentations, our research in mathematics education is hardly visible on the international scene. Hopefully change is on the way, one can see promising developments in the growing number of research projects undertaken, active collaboration with Nordic colleagues, initiating of joint international research projects, increasing number of doctoral students and so one.

Having been established on the late 1960ies Estonian didactics of mathematics has gone through the quite long period dominated by content oriented methodology- studies and only at the last decade has moved to research based didactics.

Problem areas that seem to attract most interest and research attention in Estonia at present are:

- pupils conceptions of mathematics and mathematics learning (Kislenko, Lepmann)
- teachers' and teacher students' conceptions of teaching and learning (Lepik, Lepmann,)
- teachers' professional development (Hannula, Kaljas, Lepik)
- mathematical ability, differentiation of learning (Abel, Lepik, Palu, Sikka)
- technology in mathematics education (Afanasjev, Tõnisson)
- development of mathematical knowledge (Kurvits, Lepman, Palu, Reinup).

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MATHEMATICS EDUCATION PRACTICES IN THE NORDIC AND BALTIC AREA

HOW FINNS LEARN MATHEMATICS: WHAT IS THE INFLUENCE OF 25 YEARS OF RESEARCH IN MATHEMATICS EDUCATION?

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Abstract

Firstly, I will describe the Finnish school system, the development of mathematics curricula and the implementation, i.e. working during mathematics lessons of the comprehensive school. The main part of my presentation concentrates on dissertations done on Finnish school mathematics within the last twenty five years (altogether 35 studies) as well as some established research projects, as projects financed by the Academy of Finland. At the end I will offer my personal perspectives what the development of mathematics teaching will be and what it should be.

A preliminary comment: In the paper I try to give an overview on the state-of-art in research in mathematics education in Finland. But I understand that such an attempt cannot lead to a complete view, since the presenter's basic beliefs and values form the framework in which he is able to work and form his overview. Therefore, the situation might seem different with the eyes of a researcher from another Finnish university.

Introduction

Today Finland is in the world famous as a country of excellent mathematics teaching. This acknowledgement is due to Finnish students' good success in the three consequent PISA comparisons (2000, 2003, 2006). Each time Finland has been in the group of the top three (cf. Kupiainen & Pehkonen 2008). This might be a reason why other countries are interested in our "secret weapon", i.e. how the Finnish educational system functions and what might be the reasons for our success. In order to uncover our teaching system we produced a couple of years ago the book *How Finns learn mathematics and science* (Pehkonen, Ahtee & Lavonen 2007). Furthermore, in a published paper (Pehkonen 2008) I gave background information on the development of the Finnish mathematics instruction and curricula within last 30 years. And this paper continues the same communication process.

In the first part of the paper I will describe the Finnish school system, the curriculum and its implementation. This part contains my understanding of how children work in mathematics lessons of the Finnish comprehensive school. There is not so much studies published on it. Additionally, I will sketch a vision how mathematics teaching is planned to be developed in the near future. The second part concentrates on research done in mathematics education in Finland and its consequences to school teaching. The main message of the paper is to convoy the ideas developed in the dissertations done on Finnish school mathematics within the last 25 years (altogether 35 studies). Furthermore, I will briefly introduce the most important research projects on mathematics education in the same time period in Finland.

Mathematics teaching in Finnish schools

In Finland, we have a nine-year comprehensive school that begins at the age of seven. After the comprehensive school, there are two options: the upper secondary school (grammar school) and vocational school. In the comprehensive school, mathematics is taught with 3–4 lessons per week, and in the upper secondary school there are two selective courses: advanced mathematics and general mathematics. The amount of mathematics taught in vocational schools varies according to the career, but it usually is combined with situations of the career in question.

In Finland, students in elementary teacher education take a higher academic degree (usually five years of university studies), and the subject of their master's thesis is education. The goal of the scientific elementary teacher program is to produce reflective teachers who can combine knowledge of educational sciences with knowledge of subject pedagogy, e.g. mathematical pedagogy. Mathematics teachers for secondary schools have studied at the department of mathematics about three years, when they come for their pedagogical studies of one year to our department. After that they return to their own department for a master's thesis in mathematics. For more details about the Finnish school mathematics teaching and teacher education, one may read e.g. in the published book (Pehkonen & al. 2007).

Development of the mathematics curricula

A general picture of the development of the Finnish mathematics curricula from the 1960s to around 2000 is presented in Figure 1. Changes adopted in the US curriculum played a central role in this development, with a delay of about 10 years. However, the principles of each trend were not taken as such, but they were modified in the process of implementation to better fit the Finnish educational system. For example, the objectives and contents of teaching and learning in the "back to basics" trend were deliberated thoroughly and given particular interpretations when fitted to the Finnish education system (cf. Kupari, 1994).

During the 1980s the established view on learning began to change, including mathematics teaching. Cognitive psychology, emphasising students' own construction of knowledge and learning, began to replace the older behaviouristic paradigm. Consequently, the focus of learning shifted to students' activities and to their ways of perceiving and shaping the world around them (cf. Lehtinen 1989). In the 1990s, responding to the new demand, a group of Finnish mathematics educators wrote a booklet on mathematics teaching (Halinen & al. 1991), presenting a view very similar to the later concept of mathematical literacy in PISA.

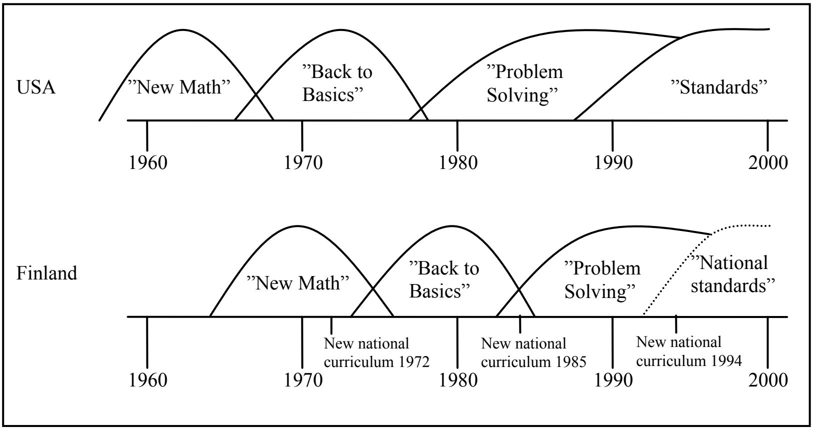

Figure 1. Development of trends in mathematics teaching in Finland and in the US (according to Kupari 1999).

Besides traditional teachers’ talk and pupils’ independent calculations, other means of teaching and learning mathematics were to be used: problem solving, exploration, discussions about mathematics, and dealing with problems rising from everyday life. In implementing these ideas, two key points arose: understanding learning as an active endeavour, and mathematics as a skill to be used and applied in diverse situations. The former meant that students should have ample time for learning and for deliberating on what they had learnt, while the latter emphasized the importance of using problems rising from everyday life. This meant tasks where the level of mathematics was not necessarily so high, but where students could apply the mathematics learnt at school in situations that were familiar and meaningful to them. For a more detailed description of the changes in Finnish mathematics teaching, see Pehkonen (2008) and Pehkonen & al. (2007).

The curricular frames

Reflecting these changes, the main objective for mathematics instruction in the Finnish curricula of 1985 and 1994 was to offer to all students basic mathematical knowledge and skills to manage in everyday life situations and later in work life (NBE 1985, NBE 1994). The developing of students' mathematical thinking and their learning of mathematical concepts and problem-solving methods were also emphasized, as well as the role of mathematics in promoting students' intellectual growth and in increasing their potential for purposeful activity and social interaction in later life.

In the 1985 curriculum (NBE 1985), content requirements were given to each grade from 1 to 9, and divided into four domains or content areas: the concept of number, expressions and equations, geometry, and applied mathematics. Focus was shifted from structure and basic concepts to application, problem solving, and everyday mathematics. In the 1994 curriculum (NBE, 1994), no specific contents were mentioned any more, and instead it stressed how the traditional content areas must be reviewed critically, and knowledge that was not necessary for understanding mathematical structures and applications was left out. Regarding the lower grades of 1 to 6, with classroom teachers, it was emphasized that students should understand the basic concepts and be able to do basic calculations mentally, on paper, and with a calculator.

In the 1985 curriculum (NBE 1985), students' own experiences, together with familiar and earlier learnt topics were taken as the starting points of teaching. The objective for stressing application and problem solving was to foster and further develop students' creativity and thinking skills on one hand, and to diversify teaching methods on the other. The spirit of it was that there should be more student-centered activities during the lessons, shifting the focus from knowing to doing. Teachers were expected to pay special attention to individual students, and to offer each one the possibility of obtaining sufficient knowledge and skills in all core content areas. Following the socio-constructive paradigm, the reformed framework curriculum of 1994 (NBE, 1994) underlined a new understanding of knowledge as one that is changing and relational with the emphasis on students' active role in constructing their own knowledge. To reflect this, learning situations were expected to be built around discussions, experiments and problem-solving, based on concrete everyday problems. Students of all grade levels should work and build models with their own hands. Calculators and computers should be used as natural aids of teaching and learning, beginning from the lowest grades. Also, mathematics instruction was to be integrated with other subjects and other work done in school.

The National Board of Education introduced new guidelines for student evaluation in spring 1999 (NBE 1999). These included, among other things, descriptions for 'good' performance (mark 8 on the marking scale from 4 'rejected' to 10 'excellent'). It was also stated that mathematics instruction must progress systematically and create a lasting foundation for the assimilation of mathematical concepts and structures. The practical hands-on nature of teaching was seen to play an important role in linking students' own experiences and ways of thinking to the abstract system of mathematics. Problems rising from everyday situations, which could be solved by mathematical thinking or operations, were to be utilised effectively.

Mathematics teaching

A typical Finnish mathematics lesson begins by checking and going through the last lesson's homework. Following this, the teacher introduces the new topic to be learnt, e.g. a new calculation method or a geometric concept, which will then be explored collectively with some examples. Then the teacher assigns students some problems from the textbook to solve individually, to make sure that everything has been understood about the underlining idea. At the end of the lesson he/she gives the students new homework from the textbook. This model was dominant in the 1980s and seems still to be so today, despite the recurring curriculum reforms (cf. Maijala, 2006; Savola, 2008). According to our experiences, though, this kind of textbook dependence is stronger in grades 1 to 6, i.e., for classroom teachers, than for the last three years of comprehensive school education with mathematics teachers.

At the end of the 1980s, the Board of General Education, the predecessor of the National Board of Education (NBE), began to publish small booklets promulgating the new constructivist understanding of learning (e.g. Lehtinen, 1989; Halinen & al., 1991 for mathematics). However, already in the early 1980s, a strong urge to revise the traditional teaching paradigm had emerged in the university departments of teacher education, with teacher in-service education focusing on new teaching methods and classroom practises. Especially during the 1980s, in-service courses were offered on the use of learning games, on problem solving, on ways to develop students' mathematical creativity, on using computers and calculators, and on constructive geometry teaching (cf. Pehkonen, 1997). In her dissertation, Tikkanen (2008), summarised the current methods used in mathematics teaching in the lower grades of Finnish comprehensive schools to three approaches: problem solving, 'real mathematics', and story telling (ibid, 96–105).

Problem solving in school mathematics

Mathematical problem solving has been generally accepted as a means for promoting thinking skills (e.g. Schoenfeld, 1985). Since the 1980s, this has been reflected in the curricular documents of most countries as an explicit or implicit emphasis on the importance of teaching students to become good problem solvers (cf. Pehkonen, 2001).

In accordance with this general trend, fostering students' problem solving skills has been one of the general objectives of the Finnish curricula now for more than twenty years (NBE 1985, 1994, 2004). However, the 1994 curriculum (NBE 1994) differed from the earlier one by providing only general guidelines, leaving their implementation mostly open with the expectation that each school would write its own, more detailed curriculum documents following these general guidelines. This was also the case for mathematical problem solving. Accordingly, success in the implementation of the general goals was left to the individual schools and teachers, relying on their professional knowledge and skills. In other words, the decentralisation of educational authority meant that teachers were given the opportunity – but also the responsibility – to participate in formulating the precise goals and methods for mathematics teaching and problem solving activities for the different grade levels, to guide and lead these activities, and to assess their outcomes.

Already in 1986, the National Board of Education (NBE) began to promote problem solving in school mathematics, arranging a seminar on the topic for teacher educators. Even if verbal tasks to be solved by equations had been a permanent part of the curriculum already before that, actual problem solving tasks had been rare in Finnish mathematics textbooks. Soon after the NBE seminar, almost every printing house published a set of problems in the form of a booklet or a deck of cards, and with time, similar problems began to appear in regular textbooks. But the share of such tasks made up about 10 % of all tasks in the Finnish textbooks for grade 7 at the end of the 1980s (Kari 1991), later (non-systematic) studies by teacher students indicate that their share has increased little since that time, despite all the discussion.

Towards the end of the 1980s, extensive in-service training was organized for both elementary and mathematics teachers in comprehensive schools to promote problem solving and new teaching methods to enhance students' active involvement in their own learning (e.g. Pehkonen 1997). And, according to Kupari (1999), in ten years both elementary teachers and mathematics teachers regarded problem solving as an important aspect of mathematics teaching. But even after twenty years, only a small part of teachers have actually changed their teaching style. Even teachers who readily express beliefs emphasizing the importance of problem solving, often fail to implement it in their own teaching (Perkkilä 2002). This demonstrates the difficulty and slowness of bringing forth a real change in teacher practises (cf. Pehkonen 2006).

In an attempt to find new methods for mathematics teaching, the open approach method, e.g. the use of open-ended problems, was developed in Japan in the 1970s (Becker & Shimada 1997). Open-ended problems, where the initial or the goal situation is not given in an exact form have been since understood to promote teaching that emphasizes understanding and creativity (e.g. Silver 1995; Stacey 1995). Students are given a free hand to formulate the problem and to choose the methods to solve it, meaning that they may end up with different but equally valid solutions and argumentations, depending on the choices made during the process.

In Finland, these ideas have been advocated since the 1980s in in-service teacher courses, in teachers' journals, and in teacher pre-service education (cf. Halinen & al. 1991). The leading idea has been to increase openness and creativity in mathematics teaching. Yet, despite all the effort, we could well borrow the words of Schroeder & Lester (1989) and say that only few Finnish teachers teach *via* problem solving even if most of them teach something *about* problem solving. In view of the strong professional education of Finnish teachers, this actual slowness of progress in implementing problem solving in mathematics classrooms raises questions. One reason seems to be classroom time management practices and assessment methods that do not favor the sustained engagement that problem solving calls for. Also, problem solving and open-ended problems do not fare well if students are afraid of making mistakes; still a common enough feature in Finnish schools.

Assessment in mathematics

There is no national testing in Finnish comprehensive schools. Yet, like in some other core subjects, the National Board of Education (NBE) implements a biennial assessment of ninth graders' curricular competence in mathematics for a system-level evaluation of education with a random sample of approximately 4500 students in 130 schools. The results are analyzed, published and discussed at the system level and look at regional, gender and in-between school differences, but no school-level data is disclosed (e.g. Mattila 2002, 2005). However, municipalities can purchase these tests to assess their students for their own evaluation purposes, allowing comparison to the national level. A comparable study of sixth graders' performance in mathematics is also implemented by the NBE every fifth year to evaluate learning results at the end of the lower level of comprehensive school (cf. Niemi 2008).

Already, long before the introduction of the NBE assessments, a nation-wide voluntary test was implemented annually by the Union of Mathematics Teachers for the students of the last year of comprehensive school (ninth grade). The results were published in the mathematics teachers' journal, so that teachers have the opportunity to compare their students' level of attainment to that of other students. However, these were rarely discussed more widely.

Mathematics education research and its influence

About 30 years ago (in 1974) in connection to the university study reform, elementary teacher program was moved from pedagogical high schools to universities. At that time eight teacher education units (Helsinki, Joensuu, Jyväskylä, Oulu, Rovaniemi, Tampere, Turku, Vaasa) were established. Typically there are a compound of department of education and department of teacher education. In some units there were several departments of teacher education, the so-called filials. For example, the teacher education unit of University Turku has two departments of teacher education, Turku and Rauma.

In this connection new positions in mathematics education were established, both for professors and for lecturers. Professor positions (as a matter of fact professorships for education of mathematical subjects) were established four: Helsinki, Jyväskylä, Oulu, Vaasa. These positions have a research obligation, and therefore, research on mathematics education got much new power.

Firstly, I will introduce the 35 dissertations made in Finland in the field of mathematics education within the last 25 years. Furthermore, a group of some larger research projects are dealt with, such as research projects financed by the Finnish Academy. Secondly, I will discuss their meaning for mathematics teaching in schools.

Dissertations

Here we concentrate on dissertations done in Finnish school mathematics within the last 25 years (altogether 35 studies). Most of them are written in Finnish, there are only five dissertations in English, and two in Swedish. The dissertations are roughly divided into six sections: learning requirements (7), teaching in elementary school (8), teaching in middle school (7), teaching in high school (4), university students (4), mathematics teachers (5). The list of all 35 dissertations is given chronological order in the appendix (the title of the dissertation is translated into English).

Learning requirements	**Teaching in elementary school**	**Teaching in middle school**
Kallonen-Rönkkö (1984)	Vornanen (1984)	Silfverberg (1999)
Aitola (1989)	Lindgren (1990)	Joki (2002)
Yrjönsuuri (1989)	Sinnemäki (1998)	Hihnala (2005)
Ruokamo (2000)	Häggblom (2000)	Törnroos (2005)
Malmivuori (2001)	Niemi (2004)	Attorps (2006)
Linnanmäki (2002)	Räty-Zaborsky (2006)	Hassinen (2006)
Hannula (2004)	Leppäaho (2007)	Näveri (2009)
	Tikkanen (2008)	

Teaching in high school	University students	Mathematics teachers
Repo (1996)	Huhtala S. (2000)	Kupari (1999)
Merenluoto (2001)	Kaasila (2000)	Huhtala M. (2002)
Joutsenlahti (2005)	Pietilä (2002)	Lilja (2002)
Hähkiöniemi (2006)	Viholainen (2008)	Perkkilä (2002)
		Soro (2002)

Here I will describe each of these dissertations with a couple of sentences. Because of space limits I am compelled to restrict my comments to the most important one.

Learning requirements

When researchers have near relations with general education, they usually concentrate in their study in learning and its requirements. Marja Kallonen-Rönkkö (1984) investigated the intellectual prerequisites for learning and their applicability in lower grades of the comprehensive school. Anneli Aitola (1989) used high school students' personal traits in order to figure out how they use strategies typical to them in learning. Raija Yrjönsuuri (1989) studied students' mathematics-related orientations in upper secondary school. Heli Ruokamo (2000) investigated pupils' problem solving skills and their development in proper learning environments. Marja-Liisa Malmivuori (2001) wrote a compendium on mathematics education research of affective domain, trying to sketch a theoretical overview. Karin Linnanmäki (2002) analyzed pupils' self-concept and its development. Markku Hannula (2004) described how affective reactions are born and developed in pupils' mind.

Teaching in elementary school

Elementary school mathematics has been an interested topic through the whole period, and the researchers have been either elementary teachers or elementary teacher educators. Irma Vornanen (1984) developed according to the principles of Piagetian developmental psychology a teaching program that aims to develop the number sense of first graders. Sinikka Lindgren (1990) concentrated on the use of hands-on materials, and on pupils' reactions of their use. Jussi Sinnemäki (1998) studied the meaning of mathematical learning games for pupils' motivation. Lisen Häggblom (2000) followed pupils' mathematical development through all grades of the comprehensive school. Eero Niemi (2004) analysed the results of a nation-wide mathematics test for sixth-graders. Sinikka Räty-Zaborsky (2006) compared the Finnish and Hungarian mathematics textbook authors' conceptions on geometry and its teaching. Henry Leppäaho (2007) implemented a teaching experiment on problem solving, combining in the projects used mathematics, mother tongue, arts and handy craft teaching. Pirjo Tikkanen (2008) investigated with the help of pictures drawn by pupils how Finnish and Hungarian pupils think and work when using the so-called "Hungarian style" of teaching mathematics.

Teaching in middle school

Mathematics teaching in the upper level of the comprehensive school (middle school) is a rather fresh domain, but there is already published seven dissertations in ten years. Harry Silfverberg (1999) investigated pupils' van-Hiele-levels in geometry, testing the theory and partly improving it. Jaakko Joki (2002) developed a mathematically strict geometry course for school. Kauko Hihnala (2005) studied the arithmetic-algebra transition in the comprehensive school. Jukka Törnroos (2005) concentrated in mathematics textbooks, and looked through them the Finnish results in the TIMSS study. Iiris Attorps (2006) had the concept 'equation' as her central idea, and investigated Swedish teachers' level of understanding on equations. Seija Hassinen (2006) has observed her own school practice, and improved her teaching strategies in algebra as well as developed a new teaching model IDEAA. Liisa Näveri (2009) compared pupils' skills within 20 years period in the upper level of the comprehensive school.

Teaching in high school

Usually teachers in the upper secondary school (high school) have been interested to choose their topic from upper secondary school. Sisko Repo (1996) used DERIVE program when teaching secondary students in school. Kaarina Merenluoto (2001) had the concept of real number as a topic of her dissertation; she investigated secondary students' conceptions on real numbers and limits. Jorma Joutsenlahti (2005) inquired secondary students' thinking skills and beliefs on mathematics. Markus Hähkiöniemi (2006) dealt with students' derivate representations and their meaning for studying school mathematics.

University students

The studies on university students are mainly concentrated on teacher students. Sinikka Huhtala (2000) investigated nurse students' mathematical skills and developed a remedial program. Raimo Kaasila (2000) followed his own elementary teacher students through interviews, observations and narratives, in order to figure out their conceptions on mathematics and their development. Anu Pietilä (2002) had as research participants her own elementary teacher students whose view of mathematics she revealed with interviews and developed a model of students' view of mathematics. Antti Viholainen (2008) was interested in mathematics teacher students' knowledge and understanding on derivative and differentiability.

Mathematics teachers

Mathematics teachers have also been an important topic in Finnish research on mathematics education. Pekka Kupari (1999) made a longitudinal study figuring out how teachers' mathematics-related beliefs have been developed during twenty years of the comprehensive school. Mikko Huhtala (2002) studied teachers' views on factors affecting outcomes in mathematics in vocational schools. Kari Lilja

(2002) investigates factors influencing outcomes in mathematics in the comprehensive school. Päivi Perkkilä (2002) dealt with elementary teachers' use of mathematics textbooks in schools. Riitta Soro (2002) compared teachers' beliefs on boys and girls as learners in the comprehensive school.

Other research projects

Here I focus on some research projects in mathematics education that have an established status e.g. by getting finance from the Academy of Finland.

A red rode in the research program of Erkki Pehkonen has been the use of open problem tasks in school; the program is a compound of three Academy projects: The first project "Open tasks in mathematics" was implemented in the upper grades (grades 7–9) of the comprehensive school in 1989–92 in Helsinki area. It was focused on how problem fields (a certain type of sequences of open tasks) could be used as enrichment of ordinary mathematics teaching and what kind of influences the use of the problem fields has (cf. Pehkonen & Zimmermann 1990). The second project "Development of pupils' mathematical beliefs" was implemented in 1996–98 in schools of Helsinki area, since in the first project teachers' and pupils' beliefs were recognized as obstacles for change (cf. Hannula & al. 1996). The third project "Teachers' conceptions on open tasks" that was implemented in 1998, concentrated on the second observed obstacles: teachers' pedagogical knowledge (cf. Vaulamo & Pehkonen 1999).

Additionally, there are two other Academy projects by Erkki Pehkonen: Research project "Understanding and Self-Confidence in School Mathematics", financed 2001-03 by the Academy of Finland, was implemented in Turku and Helsinki area. The focus was fifth-graders' and seventh-graders' mathematical understanding and their self-confidence, using i.a. the concept of infinity. The second research project "Elementary Teacher Students' Mathematics" was financed 2003–06 by the Academy of Finland. It was implanted in three universities (Helsinki, Lapland, Turku) with those elementary teacher students who began their studies in 2003.

Other research projects that were financed by the Finnish Academy were Erno Lehtinen's Pythagoras project (University of Turku), and the bigM project by Simo Kivelä (Technical University, Espoo). The first one focused on real number concept in upper secondary school (cf. Merenluoto 2001), and the second one developed virtual materials for the first-year mathematics students mainly in technical universities (cf. Kivelä & Spåra 2001).

One of other bigger and long-lasting research projects was Lenni Haapasalo's MODEM project. He began the project in the 1980's at the University of Jyväskylä, the project focused on systematic practicing of concepts in different areas of school mathematics (cf. Haapasalo 1991). One example is to teach the concept of straight line for an eigth-grader using computers (cf. Haapasalo 1994).

In Vaasa, Ole Björkqvist has developed and experimented tasks for open pupil assessment. He called his research project EMU, Effektiv matematikundervisningen, but there seems to be very scarcely papers written on it (cf. Burman 2003). Björkqvist's influence in Finland is important in the group of Swedish-speaking mathematics teachers.

Influence of research on mathematics teaching

In Finnish school mathematics, there have happened many changes within the last 20 years. Some of the changes can be connected with the dissertation mentioned. In the following, I will refer to some dissertations that have had bigger influence than the other ones, without devaluing the other one (the references for dissertations are in the appendix):

One clear change in mathematics teaching of the comprehensive school has been the use of concrete teaching materials, so-called manipulatives or hands-on materials. About 20 years ago Sinikka Lindgren (1990) published the dissertation on using manipulatives in studying mathematics; her method was to establish a mathlab that she experimented in the second grade of the comprehensive school. In the 1980's, there was a lot of in-service education for elementary and mathematics teachers of the comprehensive school (cf. Pehkonen 1997). Today especially younger teachers are used to deal with concrete teaching materials.

Within the last 20 years, the meaning of affective domain for mathematics learning has been in the focus. The affective domain contains concepts like self-confidence, beliefs and conceptions. The first dissertation in this domain was that of Sinikka Huhtala (2000). Her theme was practical nurse student's own mathematics. On her clinical interviews during the study, she wrote a booklet with the title "I disgust this math…" A student's own mathematics as an explainer for learning difficulties (Huhtala 1999). This 50-page booklet has been used about ten years in teacher education programs, both in elementary and mathematics teacher education, because it revealed very clearly results of weak teaching in students' comprehension of mathematics. Markku Hannula (2004) developed some pieces of the theory for affective domain in mathematics learning, among others, he explained how attitudes get their birth. Thus, Hannula's dissertation offers basic understanding for mechanisms of learning.

Dealing with beliefs and conceptions was an important topic in Finnish dissertations. In the beginning of the 1990's, there were two dissertations on elementary pre-service teachers' mathematics-related beliefs: Raimo Kaasila (2000) and Anu Pietilä (2002). They both revealed students' mathematics beliefs that they used in their teaching in elementary teacher education programs (in University of Lapland and resp. University of Helsinki). Kaasila used students' narratives as his main tool, whereas Pietilä concentrated in students' view of mathematics. These dissertations supported the use of beliefs and conceptions in teacher education programs, both in elementary and mathematics teacher education.

In beliefs and conceptions there are two other dissertations that have had influence on teachers' thinking and performance: Riitta Soro (2002) raised the question of gender problems in teachers' consciousness. Iiris Attorps (2006) revealed mathematics teachers' conceptions about equations, and showed that their understanding on basic mathematical concepts was very weak. Her participants were Swedish mathematics teachers, but there are not so big differences between Finnish and Swedish cultures.

Kaarina Merenluoto (2001) brought into the discussion on mathematics education a new concept 'conceptual change'. Her dissertation dealt with the enlargement of the number concept as a conceptual change in mathematics. Conceptual change has been internationally a topic for a longer time e.g. in Greece (cf. Vosnidou 1994). Since Merenluoto's dissertation Finnish teacher educators have taken into account conceptual change in their instruction.

Mathematical thinking has been a topic in Finnish in-service teacher education since the 1980's (e.g. Pehkonen 1984). Until 2005 there was published a dissertation on mathematical thinking. Jorma Joutsenlahti (2005) concentrated in characteristics of task-oriented mathematical thinking among students in upper-secondary school. The dissertation has strengthened the general discussion on mathematical thinking.

Problem solving has also been in the focus of Finnish in-service teacher education since the 1980's (cf. Pehkonen 1997). But Henry Leppäaho (2007) was the first one to select it as a topic of his dissertation. He dealt with methods of teaching mathematical problem solving skill in a Finnish comprehensive school, and designed and implemented a problem-solving course. This dissertation strengthened the meaning of problem-solving in mathematics teaching. When earlier describing mathematics teaching in the Finnish comprehensive school I borrowed the words of Schroeder & Lester (1989), and said that only few Finnish teachers teach *via* problem solving even if most of them teach something *about* problem solving.

The last important cluster of dissertations is around algebra teaching in comprehensive school; there are altogether three dissertations around the topic. Kauko Hihnala (2005) investigated the development of pupils' mathematical thinking when shifting from arithmetic to algebra in comprehensive school. Seija Hassinen (2006) experimented in her own classes an alternative way to teach algebra, and named her method idea-based school algebra. Liisa Näveri (2009) made a longitudinal study, testing pupils in algebra in the upper grades of comprehensive school with the same test as 20 years ago. Therefore, the meaning of algebra learning has been emphasized in Finnish schools.

The authors of dissertations have presented results of their studies both in Finnish teacher journals and during the in-service training days of the Mathematics Teachers' Union (MAOL).

Conclusion

Finland have ranked well in all three PISA comparisons (2000, 2003, 2006). But a closer look on the PISA results shows that the Finnish achievement level in many basic tasks of the PISA tests was only 50–70 % or less (cf. Kupiainen & Pehkonen 2008, 130). The fact that the other countries' achievements were still worse, does not make the Finnish achievement good. It only shows that the level of mathematics teaching in all countries should be raised, also in Finland.

Perspectives of development in mathematics education

Now we can ponder, to which direction and how far we are moving on a short time interval. In Finnish mathematics teaching the direction seems to be to more individualizing in the comprehensive school, and mass teaching in the secondary schools. Teachers try to balance between big teaching groups and those children who demand special attention. Even more children from one-child family are coming to school, who are accustomed to the unshared attention of their parents and who have difficulties in social relationships. Television and other media offer shortsighted varying experiences, usually always more and more shocking. With these media experiences mathematics cannot compete, since they will demand persistent working, in order to learn and to understand.

To emphasize problem-solving and self-initiatives seems to be a correct direction. But problem-solving should be used as a teaching method, and not only to solve separate problems. All new information should not be given in a "ready form", but the teacher should lead pupils through self-initiative thinking to the learning objectives. Problem posing is in a near connection to such a teaching style.

Means for the change

In order to reach such a goal, following means might be used: Especially the subject matter knowledge of elementary teachers should be strengthened. Now most of the elementary teachers' knowledge is on the calculation level, some of them not even necessary properly on it (e.g. Merenluoto & Hurme 2009). The teachers should reach in mathematics the level of conceptual understanding, and that is not the case of many teachers (cf. Merenluoto & Pehkonen 2002, Kaasila & al. 2005). Also most of mathematics teachers in the upper grades (7–9) of the comprehensive school teach rather mechanically following the textbook. They should develop their sensitivity, in order they could recognize the receiving ability of the teaching group. It is important to learn to be sensitive to their pupils' emotions, since affects have a central role in learning (cf. Hannula 2006). In connection to this, there is a demand on the development of teachers' communication readiness, special attention should be fixed on the level of pupils' listening. Teachers should reach at least the level of comprehension in listening (cf. Ahtee & Pehkonen 2005), and not only be satisfied with evaluative listening.

During the last years there is less emphasis given to the use of computers or more generally to the use of ICT possibilities in mathematics teaching (cf. Ilomäki 2008, Liiten 2008). In this domain, we should get much experimental and developmental activities that demand financial support from the government. In Finland we need new types of solutions and different kinds of thinking to help the implementation of mathematics teaching.

Now we can say e.g. in the case of problem solving in Finnish schools using the language proposed by the published paper Schroeder & Lester (1989): Most teachers are in the teaching problem solving in the first phase (teaching *about* problem solving), i.e. they deal with separate problems, mathematical puzzles, in order to develop their pupils' thinking skills. Only a few teachers are in the phase 3 (teaching *via* problem solving), i.e. using problem solving as a teaching method.

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APPENDIX: The list of the Finnish dissertations in mathematics education (since 1984)

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THE DEVELOPMENT OF NATIONAL MATHEMATICS CURRICULUM IN ESTONIA AT THE BEGINNING OF THE 21ST CENTURY

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Abstract

The following Estonian national math curriculum and school reality background development aspects are observed: national mathematics curricula in Estonia after 1989; math textbooks; national achievement tests and examinations; the teaching staff and training of mathematics teachers.

Introduction

Three aspects can be differentiated in the curriculum model: intended curriculum (national context), implemented curriculum (school, teacher and classroom context) and attained curriculum (student outcomes). Mathematics curriculum in the national context will be observed more closely in the following paper.

Before World War II, curricula which were used in Estonia were strongly influenced by those in West Europe. For example, mathematics curricula emphasized the significance of functional dependence and induced the integration of different fields of mathematics (arithmetic, algebra, geometry). In 1949, the centralised all-Soviet-Union curriculum was enforced in Estonia. Since the beginning of the 1960s, mathematics as an integral school course, similar to the pre-WWII times was re-established, and re-introduction of new textbooks of Estonian origin was achieved. In the whole Soviet Union the above-mentioned re-establishments were applied only in Estonia. The syllabus remained the same until the year 1989.

National mathematics curricula in Estonia after regaining independence

Since 1989 a number of drafts for school mathematics curricula have been drawn up in Estonia. From those, two were implemented at school: the 1996 and the 2002 curricula. The latter is also currently effective national curriculum. The 2002 mathematics curriculum (Anon1, 2002) was developed in the form of a draft based on different education-political approaches. The main principle behind those approaches was the so called *personality-based approach*. Education officials interpreted the latter primarily as a cut-down from the obligatory part of syllabuses. In that framework, the time that students spend on studying mathematics within the period from the grades I to XII was reduced from 55 to 45 hours a week. Since that time, for example, the topic *integral* has not been taught at school any more. In 2005 two new drafts were compiled along with the effective 2002 curriculum.

At the request of the Estonian Ministry of Education and Research in the autumn 2006, a subsequent draft for school mathematics curriculum was drawn up (Anon2, 2006*)*. It attempted to maintain the number of weekly hours in mathematics in compliance with the still effective 2002 curriculum (compulsory school with 36 hours a week and 9 obligatory courses in gymnasium, all in all). It is noteworthy that in both drafts of the years 2005 and 2006, in addition to the obligatory course in gymnasium, the so called extended course curriculum had been suggested with a planned volume of 15 courses. Not any of the three drafts submitted in 2005-2006 materialized, mainly for education-political reasons. Despite that the ministry has requested the new mathematics syllabus board, set up in 2008, to work out a new draft for the curriculum. As far as compulsory school is concerned, the ministry of education has set as a goal to reduce students' overall study load and coordinate it better with the time allotted to studying mathematics. In addition, compilers of curricula are advised to increase inter-subject integration and improve consideration for students'

School stage	I			II			III				High school			
Form	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**		**10**	**11**	**12**	**Total**
2009 project	3	3	4	5	4	4	5	4	4	Comp	8			**44**
										Adv	14			**50**
2006 project	3	3	4	5	4	4	4	4	5	Comp	3	3	3	**45**
										Adv	5	5	5	**51**
2005 Tallinn project	3	4	4	4	4	5	5	5	5	Comp	3	3	3	**48**
										Adv	5	5	5	**54**
2005 Tartu project	10			13			13				6			**42**
2002	10			13			13				9			**45**
1996	9-13			12-15			12-15				9			**42-52**
1993	3	4	5	5	5	5	5/4	5	5	H	3	2	2	**48,5**
										SM	4	4	4	**53,5**
										G	3	3	3	**50,5**
										IMP	5	5	5	**56,5**
1991	3	4	5	5	5	5	5/4	4/5	5	H	3	2	2	**48**
										SM	4	4	4	**53**
										G	3	3	3	**50**
										IMP	5	5	5	**56**
1989 project	3	4	5	5	5	5	4	5	5	Core	3	2	2	**48**
1989	4	5	6	6	6	6	6	6	4	Comp	3	3	-	**55**

(Comp - compulsory course, Adv - advanced course; H - humanities branch, SM - science and mathematics branch, G - general branch; IMP - the intensive classes for mathematics and physics)

Table 1: The number of hours in mathematics in various curricula or their drafts.

individual development. The ministry has set the reduction of obligatory component in the curriculum as a starting point for the development of general curriculum for gymnasium. This is why the volume of the obligatory course in mathematics has been reduced from former 9 courses (9 hours a week all in all) to 8 courses (8 hours a week). At the same time, along with an obligatory course consisting of 8 courses, it is allowed to plan a 14-course extended course. In its activity the above mentioned board has departed from the still effective 2002 curriculum as well as from all the three drafted versions, compiled in the years 2005-2006. It is a pity that the compilation of all the projects under discussion has taken and is still taking place in an indefinite education-political situation, for example, as far as the general number of hours in mathematics and their division among classes is concerned.

Textbooks

Beside the curriculum, mathematics textbooks also have an essential impact on implemented curriculum. As was said above, the textbooks compiled by Estonian authors have been in use since the beginning of the 1960s. It is remarkable that one of the Estonian mathematics textbooks for grades V-VI (Telgmaa & Nurk, 1988, 1989) was awarded the first prize at an all-Union competition in the Soviet Union in 1987 and is still in use in a number of former union republics. In the 1990s, a lecturer from Ohio University, working in Vologda at that time, got acquainted with the textbook. At his initiative the textbook was translated into English in 2003 and it is still in use in the state of Ohio in the U.S.A.

Immediately after Estonia regained its independence at the beginning of the 1990s, numerous new mathematics textbooks appeared at school. If before there was only one kind of mathematics textbook for each grade, from now on a possibility of compiling alternative textbooks was welcome. At present in each grade from I – XII, the teachers have a possibility of choice among three different mathematics textbooks. For grades IV – VI, a textbook series was translated from Lithuanian but by now it has been considerably re-worked by Estonian author (Kaasik & Balčytis, 1998; Kaasik & Cibulskaitė & Stričkienė 1999, 2001). Estonians are the authors of all other mathematics textbooks.

Since the number of mathematics lessons varies in different schools, most of mathematics textbooks contain more material than the national curriculum requires. It often happens that the teachers plan their work on the basis of the textbook in their hand and so plan to teach all the material in it even in the case of covering only minimum hours.

National achievement tests and examinations

In 1996 already the rule was laid down that actual realization of competencies established in the national curriculum or the so called attained curriculum is going to be tested at the end of each school stage. Such an external assessment is organized by the National Examinations Centre. At the end two first school stages (in grades III and VI), a national achievement test is administered every year. The examinations centre selects about 80 schools (1200–1500 students), that have to submit their tests for analysis. The selection is made on the principle that proportionally all the counties, Estonian- and Russian-medium schools as well as different types of schools were represented. Analysis-based summaries are published on the homepage of the examinations centre and are accessible for all concerned. The rest of schools may set achievement tests if they wish but practically all counties request every school to do it. Thus each county collects the results from schools for their own analysis.

At the end of grade IX all students have to take an obligatory final examination. Tasks are compiled at the National Examinations Centre.

Tests are marked at school in accordance with the evaluation manual sent to school by the examinations centre. Likewise, the examinations centre selects c. 1000 papers from among compulsory school final examination for an in-depth analysis. This analysis, too, is made public online. The examination papers include four obligatory tasks and from among the other four students have to choose two. These tasks test the acquisition of all the material learned in compulsory school.

At the end of grade XII, the national examination in mathematics is optional. While students are admitted to higher educational institutions on the basis of the outcomes of national examinations, it is important to prepare for that examination, and the process has a strong impact on the content of mathematics teaching within the last three grades. The tasks for this examination are also compiled by the board made up in the examinations centre. Examination tasks are not marked at school but in the examinations centre by an all-republican board. The examination paper consists of two parts. The first part contains six obligatory tasks, then after a 45-minutes interval, in the second part students have to solve three tasks out of four. Although the exam is based on minimum curriculum, the tasks in Part two are cognitively more complicated. The number of students who choose mathematics examination has reduced by the year (see Table with the data about last five years). It is probably related to a general unpopularity of subjects of science across the country. That is why during the recent years technical specialities at universities experience difficulties in finding competent students.

Year	2004	2005	2006	2007	2008	2009
Students taking mathematics exam	54%	52%	44%	40%	36%	37%

Table 2: The percent of the students taking national mathematics exam

Since the results of all achievement tests and examination papers are quite thoroughly analysed each year, the ministry has a detailed overview of the state of the art in mathematics teaching in the country. On the other hand, based on the papers it is possible to substantially monitor the teaching of mathematics. For example, when in 1996 the theory of probability was first included into the curriculum, a number of teachers did not teach it. The situation improved at once when a task on it was included into the examination paper.

The teaching staff and training of mathematics teachers

In its greater part, any curriculum is realised through the teachers' work. About 2000 mathematics teachers are engaged in the teaching of mathematics in schools of Estonia. The current teaching staff is characterised by a super-high proportion of female teachers (c. 90%) and the teachers' advanced age. 40 % of our mathematics teachers are over 50 years of age. At that, the most required specialists by schools are teachers of mathematics and of English. Relatively few young people come to be trained as teachers of mathematics.

In Estonia, mathematics teachers are trained at the University of Tartu (Tartu University) and Tallinn University. Tallinn University has concentrated the training of mathematics teachers for both compulsory school and gymnasium into one structural unit – the Institute of Mathematics and Sciences. At Tartu University, teachers for gymnasium (for grades V–XII) are trained in the Faculty of Mathematics and Computer Science and teachers for compulsory school (for grades V–IX) in the Faculty of Education. In the Faculty of Education the students study on the so called Curriculum of Sciences and after graduation may teach mathematics, physics and/or computer science. In Tallinn University, after graduating their BSc programme, students may continue in a Master's programme specialising as mathematics teachers either for compulsory school or for gymnasium.

In Estonia, after regaining independence, mathematics teachers have been educated mainly on three different curricula. For example, the so called transition curricula were effective at Tartu University until 1991. Since 1991, the curricula known as 4+1 were enforced: 4 years of Bachelor's programme and 1 year of teacher training. At that a year-long teacher training contained general-pedagogical subjects in the capacity of 8 CP, didactics-related subjects – 8 CP, electives on speciality or general didactics – 8 CP, teaching practice – 10 CP and graduation paper on teaching – 6 CP. Respective data about Tallinn University are found in Tiiu Kaljas' article in this book.

Beginning with 2002, both Tartu University and Tallinn University adopted the 3+2 system: 3 years of BSc studies and 2 years of MSc studies. Respective curricula with only a few definite changes are effective also at present. In the framework of BSc studies, students obtain speciality-oriented basis in mathematics. Master's studies at Tartu University enable students either to train to become teachers or continue in-depth studies in mathematics. On the other hand, at Tallinn University the MSc studies are monitored to obtaining the teacher's qualification only. At Tartu University, the teacher training Master's curriculum contains general-pedagogical subjects in the capacity of 16 CP, subject-related didactics – 8 CP, subjects related to school mathematics – 12 CP, subjects related to computer and information technology – 4 CP, teaching practice – 10 CP, Master's exams – 10 CP and elective subjects – 20 CP. The latter enables students to obtain a minor speciality, ordinarily to qualify as a computer teacher. At Tallinn University the respective figures are as follows: general-education-scientific and psychological subjects in the capacity of 21 CP, subject-related didactics – 9-13 CP, teaching practice – 10-12 CP, subject studies – 13-19 CP, Master's thesis and a respective seminar – 13 CP and elective subjects – 8 CP.

Each major change into the teacher training curriculum has brought about the reduction in the number of graduates. For example, when in 1993 36 students graduated from Tartu University with the qualification of the teacher of mathematics then in 1996 (transition to the 4+1 curriculum) only 8 students qualified as teachers. A similar influence can be observed in the case of the transition to 3+2 system – in the spring 2008 only 2 students who had studied on the curriculum qualified as teachers of mathematics at Tartu University. Within the years 1996–2008, 8-12 students a year on an average graduated from Tartu University on the curriculum of mathematics teachers for gymnasium. At Tallinn University the respective figure is somewhat higher, thus in the spring 2008 it was 16, on the 3+2 curriculum.

Summing up, it should be admitted that unfortunately graduates of Estonian universities at present cannot cover the need for mathematics teachers in the country. A large number of class teachers teach mathematics on the compulsory school level.

Conclusions

Improvement of mathematics curriculum has brought along both positive and negative features.

Positive features include

Schools themselves have the possibility of establishing the number of hours in mathematics, only the minimum number of hours is centrally required in each subject.

Each grade has a variety of textbooks and the teacher can choose an appropriate one.

The curriculum is in a relatively good accordance with internationally recognizable content of school mathematics. It is demonstrated by Estonian students' good results in international investigations such as TIMSS and PISA.

Negative features include

In the course of reorganization, the number of obligatory hours in mathematics has considerably reduced. Together with it, bulky textbooks and a strict external testing by the ministry cause severe stress in teachers and students.

Development of mathematics curriculum has taken and is taking place against the background of continually indefinite education policy. Mostly the curriculum developers have no conclusive clarity even about the number of hours allocated to mathematics and the relationship between the course and school-leaving examinations, to say nothing about continually changing requirements set to the presentation and layout of the syllabus. The request of mathematicians to teach two courses of different levels in grades X-XII (extended and compulsory) and based on those, to administer two-level examinations, too, has not been conclusively complied with yet.

The teaching staff is getting older and more female, the number of unqualified teachers is growing, in particular in compulsory school where mathematics is taught by teachers without respective education. A nationally regulated in-service system for teachers is lacking.

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QUO VADIS? - LATVIAN SCHOOL MATHEMATICS

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Abstract

The article deals with problems and addresses issues as well as offering solutions concerning the new standard in mathematics for secondary school, appropriate teaching aids and state examination in mathematics.

Introduction

In Latvia there are long-standing traditions for teaching mathematics in schools, and preparing secondary school mathematics teachers in universities. To become a secondary school mathematics teacher one should take the five-year university course and obtain the qualification of a teacher. Teachers having university education but lacking the corresponding qualification are given the opportunity to undertake a correspondence course (lasting 2.5 years) and obtain the needed qualification without interrupting their work. It should be noted that the duration of teaching practice has been changed from 12 weeks (9 ECT) to 26 weeks (39 ECT). That practice period was divided into three placements providing the opportunity to spend the last placement period at the university associating it with the development of one's diploma paper.

Also, the work with gifted pupils brought positive achievements. Here the greatest contribution is made by the Correspondence Mathematics School (NMS) of the University of Latvia under the management of professor A. Andžāns.

Certain novelties can also be noted in the field of general secondary education. Namely, a new standard in secondary mathematics (grades 10-12, students of age 16-18) came into force in 2008. This standard was approved by the Cabinet of Ministers and its development was facilitated by the ESF project "Natural Sciences and Mathematics" where under the management of dr.paed. D. Namsone, the best teachers of the Republic in cooperation with the educationalists from the universities made radical changes in the teaching philosophy.

Centralized state examinations for grades 9 and 12 have been introduced.

Topicality of the problems

The article addresses three issues. Firstly – does the standard in mathematics comply with the latest findings in didactics and can the goals it sets be reached? Secondly – how can the acquisition of the standard be ensured? Thirdly – can the main requirements of the standard be tested by the centralized examination and what result has already been achieved?

Explanation of the problems

The general secondary education subject standard determines that the aim of school mathematics is to improve the skills of using mathematical methods for world cognition and diverse activity by expanding the understanding of the role of mathematic models in describing the processes of environment and society and developing skills of mathematical reasoning. It is intended that the subject "Mathematics" shall empower the student to improve the understanding of the diversity of mathematical models and skills to use them, improve communication and cognition through problem solving, judgement conclusion and argumentation, as well as improve the understanding of the importance of mathematics in everyday life and its role in the development of other sciences, society and each individual. It can be observed that the emphasis has shifted from clearly formal acquisition of knowledge to its conscious application.

One of the novelties is the fact that the subject curriculum in mathematics introduces two new competences – scientific enquiry and scientific aspect of human, social and environmental interaction within the frameworks of which students should develop and improve their mathematical cognition, communication, cooperation and practical application skills; recognizing results of mathematics as science and value of its methods. It should be particularly stressed that the compulsory subject curriculum in respect to scientific enquiry also includes:

-formulation of a research problem;

-formulation of hypothesis in the inductive manner by noticing regularities, generalizing;

-the choice or formation of a mathematical model corresponding to the problem;

-result analysis; the credibility and conformity to the context of the results obtained during problem solving;

-statements (axiom, theorem, mark, quality, definition);

-judgement conclusion, types of judgement making (empirical, inductive deductive);

-structure and nature of evidence; different types of evidence; search of information in different resources on the subject of fractals;

-evaluation of sufficiency and validity of information by applying statistical data;

-noticing analogies and tendencies; generalization, formulation of hypothesis in the exercises of combinatorics and probability.

However, fair stipulations are not enough. Despite the wide range of teacher support materials provided by the ESF project “Natural Sciences and Mathematics” schools are also in a need for good textbooks. The greatest shortage of books can be observed at the secondary level. For the elementary school the situation is satisfactory, however, the primary school is cluttered up with colourful, yet low-quality literature.

It is clear that the transition from the realisation of the old standard to the new one takes time. Therefore, certain concern has been expressed in connection with introduction of standard based centralized examinations and its results. Centralized examination is a specific procedure designed in accordance with certain regulations in order to determine person's knowledge, skills and abilities. This allows us to evaluate students' achievement in respect of requirements of the state standard, and compare students' performance (evaluation levels). Examination consists of three parts. The first part includes 25 exercises for testing one's knowledge and basic skills. Time provided for completion of the first part is 50 minutes. Maximum evaluation for this part is 25 points. The second part includes 9 exercises testing the application of one's knowledge and skills in standard situations. Maximum evaluation for this part is 40 points. The third part includes 3 exercises testing the ability to apply knowledge and skills in practical and problem situations. Maximum evaluation for this part is 15 points. Time provided for completion of the second and third part is 150 minutes. Specific weight of each part is 30% for the first part, 50% for the second part and 20% for the third part. The relation of the exercises of algebra and geometry is 7:3. Time provided for the completion of the whole examination is 200 minutes. In the following the last three exercises (ex. 10, 11 and 12) of the centralized examination are presented.

Ex. 10. *Certain amount of pellets was provided for the heating of a dwelling house. Each day the same amount of pellets is used for the heating. After the day long heating the k amount of pellets is left from the initial amount. After the b day ($a \neq b$) long heating the p amount of pellets is left from the initial amount. How many kilos of pellets were there initially?*

Ex. 11. *Justify that the equation* $\sin^2 x \cdot \cos x = 1$ *has no solutions.*

Ex. 12. *There is x black and y white balls in a dish. The probability of withdrawing a black ball is lesser than* $\frac{3}{5}$. *Jānis added 2 more black balls into the dish. Now, the probability of withdrawing a black ball is greater than* $\frac{2}{3}$. *Calculate the initial number of black and white balls! (All the balls in the dish have equal probability of getting withdrawn)*

Student performance is shown in table 1 which indicates the overall results in the country for the parts 2 and 3 of the examination.

		Boys		Girls		Country	
		Mean	N	Mean	N	Mean	N
	Ex 1	46,4227	11666	51,7820	13777	49,3246	25443
	Ex 2	41,3381	11666	31,5222	13776	36,0231	25442
	Ex 3	50,5086	11666	53,4768	13777	52,1158	25443
	Ex 4	28,9442	11666	31,4383	13777	30,2947	25443
	Ex 5	57,1990	11666	61,3389	13777	59,4407	25443
	Ex 6	37,6018	11666	44,0762	13777	41,1076	25443
	Ex 7	30,3446	11666	33,5898	13777	32,1018	25443
	Ex 8	26,1608	11666	29,3817	13777	27,9049	25443
Part 2	Ex 9	26,4112	11666	28,6633	13777	27,6306	25443
	Ex 10	11,4071	11666	12,1879	13777	11,8299	25443
	Ex 11	8,4776	11666	9,5255	13777	9,0451	25443
Part 3	Ex 12	5,2423	11666	5,5523	13777	5,4101	25443

Table 1: Overall results of the centralized examination, 2009

Conclusions

The new standard in mathematics complies with the novelties in didactics (Leuders, 2003; Hogben, 2004; Kordos, 1999; Clarke et al, 2004) and if implemented our students shall not look bad on the

European background. Better results may be achieved if corresponding methodological means will be provided. There is a lot to accomplish in this respect. The results of the centralized examination are not bad, especially taking into consideration that some of the exercises were already slightly unconventional. Content of the centralized examination does not fully cover (and is it really possible?) the requirements of the standard that are not of formal nature. It is also positive that examination includes exercises implementing the aspects of investigation and human, social and environmental interaction.

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MATHEMATICAL EDUCATION IN LITHUANIA

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Mathematical education has been changing and developing in Lithuania as everything else for the last twenty years of independence. The changes affected curriculum, attitude towards the subject of mathematics itself, teaching methodology, textbooks etc.

General Curriculum approved by the Minister of Education and Science of the Republic of Lithuania defines the core mathematical teaching content for general secondary school. After the restitution of Lithuania‘s independence the first General Curriculum draft was developed in 1994. It outlined the general aims of mathematical education:

- to ensure the possibility for students to acquire mathematical thinking typical of peoples‘ mathematical culture and necessary for the development of personality and living in the modern world;
- to make students interested in mathematics and develop their mathematical skills.

Following the new educational challenges – rapid social and economic development and improving expanded possibilities for the implementation of educational innovations, the General Curriculum is constantly improved and revised according to the above indicated aims.

The General Curriculum for mathematics is influenced by the evolution of Lithuanian society, achievements of students as reflected in the international and national research, and the needs of higher education institutions, pedagogic community and parents.

The following Acts influenced the renewal of the General Curriculum:

- *Regulations of National education strategies 2003-2012 (Seimas of Republic of Lithuania, 4 July 2003, Nr. IX–1700);*
- *Recommendation of the European Parliament and of the council on key competences for lifelong learning (2006/962/EB);*
- *National program of harmonious Education development for years 2007-2015. (Decision of Government of Republic of Lithuania, 2 October 2007, Nr. 1062).*
- *Data from International and National Surveys of Students' Achievement –TIMSS, PIRLS, PISA, CIVIC (2003-2007);*
- *International experience (e.g. Finland, UK, Canada, Australian, New Zealand, Poland etc.);*
- *Proposals of teachers, educationalists, politicians and other social partners.*

Chronology of Curricula for Lithuanian Schools:

Year 1997 – certified General Curriculum for basic schools, grades 1-10;

Year 2002 – certified General Curriculum and the Standard of Education for upper secondary schools, grades 11-12;

Year 2003 – General Curriculum and Education Standards: pre-school, primary and basic level.

Year 2008 – Educational Curriculum for primary and basic levels (grades 1-10).

In 2007-2008 the General Curriculum and Education Standards for primary and basic schools were revised. One of the main objectives of this revision was the need to focus the education content on the development of students'general and basic specific competences. The education process is also oriented towards this aim. It should involve all students into active learning as the focus is on expected learning outcomes.

Educational curricula are created, updated and implemented in Lithuania following these guidelines for the education content:

- make the content of education develop general and basic skills with special attention to training learning ability;
- deepen the individualization of learning regarding different needs of pupils;
- strengthen the attainability of the education content seeking the pupils to understand and apply with creativity what they learn;
- balance the extent of the content of subjects excluding the elements of the content that lost their relevance and including new, relevant elements.

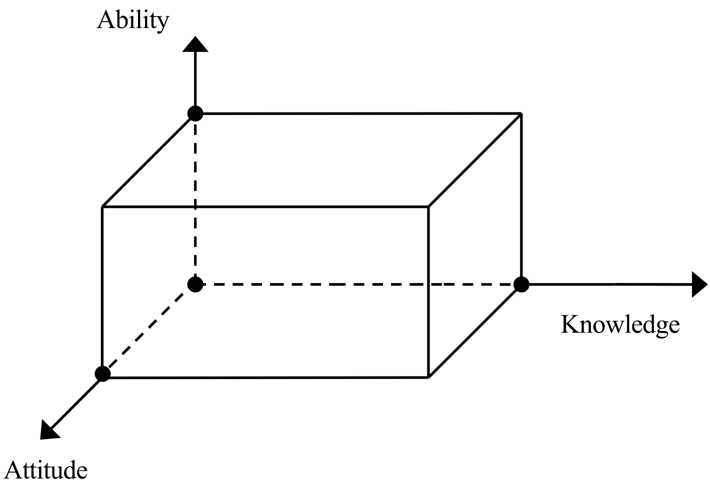

Figure 1: 3-dimensional competence model

Knowledge. Mathematical facts obtained by students which could be applied in various fields of mathematics and life.

Ability. Ability to learn mathematics and use mathematical concepts and relations for simple practical problems and in communication and collaboration.

Attitude. Understanding of historical developments, influence on modern science, practical importance and adaptability of mathematics.

Competence is ability to succeed in achieving the goal in some field which is based on knowledge, understanding, abilities and attitudes.

These curricula were put into effect from September 1, 2009, for the grades 1, 3, 5, 7 and 9 of all Lithuanian primary and secondary schools.

The structure of mathematical competence of basic level (grades 5-10) can be described by seven areas of activities:

- Numbers and calculations,
- Expressions, equations, inequalities and systems,
- Relations and functions,
- Geometry,
- Measures and measurement,
- Statistics,
- Probability theory

and five concepts describing abilities and attitude:

- Knowledge and understanding,
- Mathematical communication,
- Mathematical thinking,
- Problem solving,
- Ability to learn and interest in mathematics.

Currently the project of General Curriculum for secondary level is being prepared. The project proposes two elective courses: the general course of mathematics and the extended course of mathematics (for all Lithuanian schools that follow the compulsory education curriculum). Two Lithuanian gymnasia (Vilnius Lyceum and Šiauliai Didždvaris gymnasium) offer studying mathematics as a part of the International Baccalaureate diploma program.

The General Education Plan, approved by the Minister of Education and Science, is released once in one or two years. It defines national standard for **minimal weekly number** of lessons.

Schools are allowed to increase that number according to the students' needs and school possibilities.

Grade / **Schoolyear**	**5**	**6**	**7**	**8**	**9**	**10**	**11-12**	
							B	**A**
2002-03 P	4-5	4-5	4-5	5	3-4	3-4	5	9
2003-05 P	4-5	4-5	4-5	4	3-4	3-4	5	9
2005-07	• 8		• 8		• 7		•	•
2007-08	• 8		• 8		• 7		•	•
2008-09	• 8		• 8		• 7		•	•
2009-10	• 8		• 8		• 7		•	•

P- specialized learning, A - extended level, B- comprehensive level.

Table 1: Number of weekly lessons

Achievements of learning mathematics in Lithuanian schools are estimated in two ways: by external maturity assessment (school exam, state exam) and by national (for grades 4, 6, 8, 10) and international (PISA, TIMSS) statistics.

The assessment of students' mathematical achievements takes place at the end of basic school but it is not compulsory. After finishing upper secondary school students can choose a maturity examination of mathematics of two types: school or state exams.

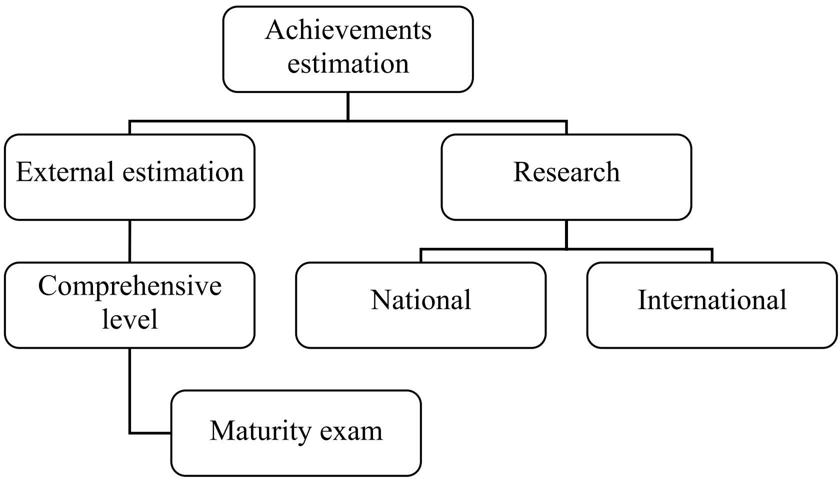

Figure 2: Assessment of students‘achievements

The maturity exam of mathematics is not compulsory; nevertheless the National Examinations Centre statistics shows that this examination is the most popular among all elective examinations.

Year	**2008**	**2007**	**2006**	**2005**	**2004**	**2003**	**2002**
Admitted to exam	13 874	14 808	16 031	17 980	17 719	16 546	14 371
Did not participate	199	188	178	389	307	266	324
Number of confirmed works	13 675	14 618	15 653	17 583	17 410	16 278	13 978

Table 2: Participation in the National mathematics exam in 2002-2008

In 2009 48 445 students participated in the maturity examinations session and 35 thousand of them took the maturity examination of mathematics.

The textbooks for mathematics are published by two main publishers Sviesa and TEV. In recent years TEV publishers offered digital mathematics and information technologies textbooks as well as mobile interactive computer books.

The quality of textbook is ensured by independent experts who assess the drafts of new textbooks. The textbooks are finally approved by the Textbook Experts‘Commission. The textbook evaluation is carried out according to the following criteria:

- accuracy of content;
- accuracy of concepts, theories, interpretations;
- validity of content;
- tolerance;
- conformity to curricula and standards;
- possibility to differentiated learning;
- validity of pedagogic system;
- possibility to use textbook on one’s own;
- diversity of visual material;
- the style of language;
- orientation to practical experience and the context of life;
- conformity to other textbooks in the field;
- design, quality of printing.

The majority of textbooks for mathematics comprise a set of learning and teaching materials; students’ book, teacher’s book, task-book, activity-book, computer-based material. There are at least three alternative sets for each class to be chosen according to students’ and teacher agreed choice.

TEACHING AND LEARNING MATHEMATICS: RESEARCH PAPERS

COUNTEREXAMPLES IN MULTIVARIABLE DIFFERENTIAL CALCULUS

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Abstract

The authors present give a short review of their experience in teaching multivariable calculus and present certain type examples (counterexamples). Counterexamples considered in the paper refer to three topics of differential calculus: limits, continuity and differentiation.

Introduction

The role of examples in learning and teaching mathematics cannot be overestimated: giving a great number of examples allows students to understand the studied concepts and relations between them better, to master the methods of solving tasks and applying them to practical problems. Our experience in teaching calculus shows whether and to what degree it is helpful for students studying multivariable differential calculus to consider certain type examples (counterexamples), which illustrate a difference from the single variable case.

Unfortunately, in a lot of textbooks too much attention is paid to the idea of cross-section of multivariable functions by varying one variable independently of others at a time while keeping the others fixed. It is a good way to study multivariable functions, thus obtaining functions of one variable, but on the other hand, this leads to failure to understand the distinction between properties of multivariable functions, executed with respect to all variables, and properties, executed with respect to each variable independently.

Counterexamples considered in the paper refer to three topics of differential calculus: limits, continuity and differentiation. In spite of all examples being considered for functions of two variables in the neighbourhood of point $(0,0)$, they can also be easily adapted to the more general case of functions of more than two variables. One can find the analysis of some of the given examples in literature: $f_1, f_3, f_7, f_8, f_{12}, f_{13}, f_{14}$ (Ляшко, 1977), f_1, f_4, f_5, f_6, f_7 (Гелбаум, 1967), f_6, f_{11}, f_{21} (Бутузов, 1988).

Limits of functions of two variables

While the notions of limit and continuity look formally the same for functions of one and many variables, they are somewhat more subtle in the multivariable case. The reason for this is that we can approach a point on the line from just two directions (left or right) but in the space there is an infinite number of ways to approach a given point. It can be shown by means of examples that the value of a limit can be dependent on the choice of this direction: $\lim\limits_{\substack{x\to 0\\ y=kx}} f(x,y)$ depends on k or $\lim\limits_{\rho\to +0} f(\rho\cos\phi, \rho\sin\phi)$ depends on φ. Then the double limit does not exist. Moreover, the existence of the unique value for limits along each line through a point does not guarantee that a function has the limit. Such situation is true, for example, for function $f_1(x,y) = \dfrac{x^2 y}{x^4 + y^2}$.

It is easy to see that

(a) $\lim\limits_{\rho\to +0} f_1(\rho\cos\phi, \rho\sin\phi) = 0$,

(b) $\lim\limits_{\substack{x\to 0\\ y=kx}} f_1(x,y) = \lim\limits_{\substack{y\to 0\\ x=ky}} f_1(x,y) = 0$,

but $\lim\limits_{\substack{x\to 0\\ y=x^2}} f_1(x,y) = \dfrac{1}{2}$. It implies that double limit $\lim\limits_{\substack{x\to 0\\ y\to 0}} f_1(x,y)$ does not exist. It also means that the convergence in limit (a) is not uniform with respect to $\varphi \in (0,2\pi)$ and the convergence in limits (b) is not uniform with respect to $k \in R$. The uniform convergence in limit (a) or in limits (b) would guarantee the existence of the double limit.

The iterated limits $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ exist but the double limit $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist for the following functions:

$$f_2(x,y) = \frac{x - y + x^2 + y^2}{x + y}, \quad f_3(x,y) = \frac{x^2 y^2}{x^2 y^2 + (x-y)^2}.$$

Since $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f_2(x,y) = 1$, $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f_2(x,y) = -1$, by this example it is proved that an interchange of the iterated limits in general is not valid. Otherwise, the existence of the unique value for iterated limits $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f_3(x,y) = \lim\limits_{y\to 0}\lim\limits_{x\to 0} f_3(x,y) = 0$ does not guarantee that the function has the double limit.

Let us consider also examples of double limits in the case when one or both of iterated limits do not exist. For function $f_4(x,y) = x + y\sin\frac{1}{x}$ limit $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f_4(x,y)$ does not exist, but $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f_4(x,y) =$ $= \lim\limits_{\substack{x\to 0\\ y\to 0}} f_4(x,y) = 0$. For function $f_5(x,y) = x\sin\frac{1}{y} + y\sin\frac{1}{x}$ both iterated limits do not exist, but $\lim\limits_{\substack{x\to 0\\ y\to 0}} f_5(x,y) = 0$. By means of these examples it is proved that the existence of the double limit does not imply the existence of the iterated limits.

Continuity of functions of two variables

Continuity of multivariable functions in general cannot be proved by using single variable methods: continuity of a function at a point with respect to each variable independently is not enough to ensure continuity of this function at a point.

Function $f_6(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$ is continuous at $(0,0)$ with respect to x and with respect to y, but it is not continuous at point $(0,0)$ along the line $y=kx, k\neq 0$.

Function $f_7(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$ is continuous at $(0,0)$ along all lines $y=kx$ and $x=ky$, but it is not continuous at this point.

Differentiation of functions of two variables

The relations between partial derivatives and directional derivatives, differentiability and continuity will be discussed in this sub-section. In particular, by means of examples it can be shown that the existence of all partial derivatives at a point does not guarantee that a function is continuous or is differentiable there. It illustrates a difference from the single variable case, where differentiability implies continuity, and the term differentiable means that the derivative exists. In the multivariable case we use the much stronger sufficient condition of differentiability - continuity of all partial derivatives; however there exist differentiable functions which do not have continuous partial derivatives.

By using the following denotations:

D - function f is differentiable at point $(0,0)$,

DX (DY) - partial derivative $\frac{\partial f}{\partial x}(0,0)$ ($\frac{\partial f}{\partial y}(0,0)$) exists,

C - function f is continuous at point $(0,0)$,

CX (CY) - f is continuous at $(0,0)$ with respect to x (to y),

DC - partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $(0,0)$,

we construct the scheme of relations between the before-mentioned properties:

$$\mathrm{DC} \Rightarrow \mathrm{D} \Rightarrow \begin{cases} \mathrm{DX} & \Rightarrow \quad \mathrm{CX} \\ \mathrm{D}Y & \Rightarrow \quad \mathrm{C}Y \end{cases}$$

$$\mathrm{D} \Rightarrow \mathrm{C} \Rightarrow \mathrm{CX}\ \&\ \mathrm{CY}$$

The below considered functions show that none of the pointers of the scheme could be reversed vice versa (it means that none of implication signs could be replaced with a sign of equivalence):

$f_8(x,y) = \sqrt{|xy|}$, $\qquad f_9(x,y) = |x| + |y|$, $\qquad f_{10}(x,y) = \sqrt[3]{x^2 y^2}$,

$$f_{11}(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0, \end{cases}$$

$$f_{12}(x,y) = \begin{cases} \dfrac{x^3 y}{x^6 + y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0, \end{cases}$$

$$f_{13}(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

$$f_{14}(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

Table 1 illustrates the role of each function in the analysis of this scheme. For example, the location of function f_8 in row C and in column ¬D means that function f_8 is continuous at $(0,0)$, but is not differentiable.

	¬C	¬ DX , ¬DY	¬D	¬DC
C		f_9	f_8, f_9, f_{11} f_{13}	f_{10}, f_{14}
DX, DY	f_6, f_7, f_{12}		f_8, f_{11}, f_{13}	f_{10}, f_{14}
D				f_{10}, f_{14}
DC				

Table 1: Properties of the functions

It is helpful for students to analyse properties of functions f_{15}, f_{16}, f_{17}, f_{18}, f_{19} and f_{20}, which are defined by using natural parameters k, m and n. Let us note that: 1) if $m=1$, then $f_{15}=f_9$;

2) if m is even, then $f_{15}=f_{16}$ and $f_{19}=f_{20}$;

3) if $m=n$, then $f_{15}=f_{20}$ and $f_{16}=f_{19}$;

4) if k and m are even, then $f_{17}=f_{18}$.

$f_{15}(x,y)=\sqrt[m]{\lvert x\rvert^m+\lvert y\rvert^m}$	if m is odd	¬ DX & ¬DY (¬D)
$f_{16}(x,y)=\sqrt[m]{x^m+y^m}$	if m is even	¬ DX & ¬DY (¬D)
	if m is odd, $m\neq 1$	DX & DY & ¬D
	if $m=1$	D (DX & DY)
$f_{17}(x,y)=\sqrt[n]{x^k y^m}$ if n is odd	if $k+m>n$	D (DX & DY)
	if $k+m\leq n$	DX & DY & ¬D
$f_{18}(x,y)=\sqrt[n]{\lvert x\rvert^k \lvert y\rvert^m}$	if $k+m>n$	D (DX & DY)
	if $k+m\leq n$	DX & DY & ¬D
$f_{19}(x,y)=\sqrt[n]{x^m+y^m}$ if n is odd	if $m>n$	D (DX & DY)
	if $m<n$	¬ DX & ¬DY (¬D)
$f_{20}(x,y)=\sqrt[n]{\lvert x\rvert^m+\lvert y\rvert^m}$	if $m>n$	D (DX & DY)
	if $m<n$	¬ DX & ¬DY (¬D)

Table 2: Analysis of the properties in dependence of parameters

Mixed partials of functions of two variables

Now we consider the notion of mixed partials $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$. It is known that the existence of continuous mixed partials at $(0,0)$ implies the equality $\frac{\partial^2 f}{\partial x \partial y}(0,0)=\frac{\partial^2 f}{\partial y \partial x}(0,0)$. Moreover, the continuity of only one from mixed partials guarantees that the last equality is true. Taking into account that for function

$$f_{21}(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0, \\ 0, & x^2+y^2=0, \end{cases}$$ we have $\dfrac{\partial^2 f_{21}}{\partial x\partial y}(0,0)\neq\dfrac{\partial^2 f_{21}}{\partial y\partial x}(0,0)$,

by means of this example we can illustrate the importance of continuity for the equality of mixed partials.

The next examples show that the equality of the mixed partials at $(0,0)$ does not imply the continuity of $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$ at this point.

For functions

$$f_{22}(x,y)=\begin{cases} \dfrac{\sqrt[3]{x^4y^4}}{x^2+y^2}, & x^2+y^2\neq 0, \\ 0, & x^2+y^2=0, \end{cases} \quad \text{and} \quad f_{23}(x,y)=\begin{cases} \dfrac{\sqrt[3]{x^5y^5}}{x^2+y^2}, & x^2+y^2\neq 0, \\ 0, & x^2+y^2=0, \end{cases}$$

the mixed partials exist and are equal at point $(0,0)$, but they are not continuous at $(0,0)$. Let us note also that function f_{22} is not differentiable at $(0,0)$, but f_{23} is differentiable at this point.

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THREE COLOUR PROBLEM AND ITS USAGE IN WORK WITH PUPILS AND STUDENTS

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Abstract

The different problems involving V-trominoes (known as "corners" in mathematical Olympiads) and some findings of pupils and students are considered. A few challenging problems appropriate for pupils research are offered. Historical information is given as well.

Introduction

The Title *Three Colour Problem* is taken from the web (*Vee-21*). The main *hero* of this paper is a shape made up of three equal squares and known as V-tromino (Alternate names: L-tromino, L-triomino, V-triomino), see Figure 1. From time to time this shape is explored in mathematical Olympiads problems. My mathematical puzzle (abbreviated as V-27) was elaborated especially for the presentation in the 28th *International Puzzle Party* (IPP28), Prague, 2008. V-27 was manufactured by Anatolijs Pungins (Riga, www.lazeroptima.lv). It has been made from laser cut plastic and consists of 27 pieces in three colours (9 in red, 9 in green, and 9 in blue), the tray 9 × 9, and a set of 28 problems to be solved. The short descriptions, colour photos, solutions (one page) of V-27 and 99 other exchange puzzles is the main content of the book (28 International Puzzle Party ,2008). V-27 allows dozens of exercises of varying difficulty, from straightforward to Olympiad-calibre and unsolved problems. This polyfunctional puzzle is very appropriate in work with pupils. *Three Colour Problem* and/or its generalizations have been investigated by pupils and students in their contest or research papers (Balode & Petrov*a,*2009*;*. Kļimova 2004*;* Pūre 2009). In fact, the idea to elaborate V-27 came into my mind from the puzzle Vee-21 (see *Vee-21*). V-27 has some advantages: assembles of the square 9 × 9 do not contain holes; more problems and more difficult challenges; new themes for research. For more information it is worthwhile reading *The Tromino Puzzle* by Norton Starr (2009) containing 24 references and a product of *Kadon Enterprises* (*Vee-21, 2003)* where 28 problems for barricaded grids with V trominoes have been included. Some problems from the brochure (*Vee-21, 2003)* was a starting point of bachelor's paper by E. Pūre (2009). Further such notions are used:

Polyomino – a connected array of identical squares having the property that any two squares do not touch or else meet along an entire, common edge; *n-mino* – polyomino consisting exactly of *n* identical squares. *Contact point* – a point where trominoes of equal colours touch each other. *Contact point of type* 1-1-2 – a point where four trominoes touch each other and two trominoes have the same colour (Figure 2). *Contact point of type* 2-2 – a point where 2 pairs of the same colours touch each other (Figure 3) V-*polygon* – a polygon assemblable from V-trominoes. Find a *solution*, i.e. fill (without overlapping) the given tray by V-trominoes.

The main task of the *Three Colour Problem* is to fill the tray by V-trominoes so as to minimize the number of contact points. The source (*Vee-21*) with intriguing questions: "Can the pieces of the same color be separated so that they don't meet even at corners? Doing this with one color is easy. Total separation of two colors is very hard. All three colors?" serves as a great stimulus to elaborate the contest paper (Klimova, 2004). As stated by a programmer Atis Blumbergs (Latvia, 2003) the puzzle Vee-21, (see *Vee-21*) has 8 optimum solutions (having only one contact point) and only one of them has a contact point of the type 2-2. He has also analysed rectangles 9 × 9 and 9 × 10. Three is the minimum number of contact points for both these rectangles. Moreover, for 9 × 9 there is a nice solution, namely, each colour meets exactly in one point. Such a solution is unique with a precision to rotations and colouring. As far as I know the problem of minimizing the number of contact points is not solved for larger squares 12×12, 15×15, etc.

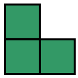

Figure 1

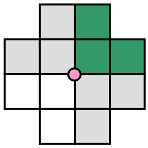

Figure 2

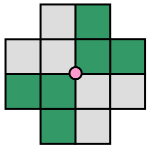

Figure 3

V-trominoes in work with pupils and in pupils' or students' papers

How to use V-trominoes in work with pupils? One of the possibilities is looking through literature and selecting problems involving these shapes. For example, Balode and Petrova (2009) ran through 4,070 problems of Latvian mathematical Olympiads (1973-2008) and found 20 problems of the required type. The second possibility is to create a set of needed problems oneself. I recommend these possibilities, however, preferring the latter one. In general the answer to the question raised depends on several factors. It may depend on teachers', students' or other persons' aims, a level of the mathematical background, possibilities to use only pen and paper, chalk and blackboard or some modern technologies; it would be fine to have a puzzle for each pupil. We may be asked to conduct an extracurricular lesson for the whole class, or to train systematically some gifted pupils in solving Olympiad problems, or to be a supervisor of pupil's research. It is understood that we will explore a different set of exercises, methodology, methods in various situations. It is advisable to begin the work with pupils offering them simple exercises. Let us consider a simple, however, non-trivial and didactic problem: *Mark all the unite squares of the square* 5×5 *so that the remaining part is tileable by V-trominoes* which occasionally appears in mathematical competitions: *The square* 5×5 *is divided into*

25 *equal cells*. *One of the cells is removed*. *Can the remaining region always be cut in* 8 *corners*? (*Corner* = V-tromino). My experience shows that several pupils at first try to solve this problem in an unreasonable way checking all the cells. A few pupils soon observe that due to symmetry their work is reduced to checking 6 cells. The solution is easy to be found for 3 of these cells, see Figure 4. As to the remaining 3 cells pupils believe in the non-existence of solution, but as a rule they do not know how to prove it.

To obtain a short proof let us mark 9 cells as in Figure 5. Assuming one of the 16 unmarked (white) cells is removed, 9 trominoes are necessary for covering 9 marked cells. But we have only 8 trominoes. Thus, none of 16 white cells can be removed.

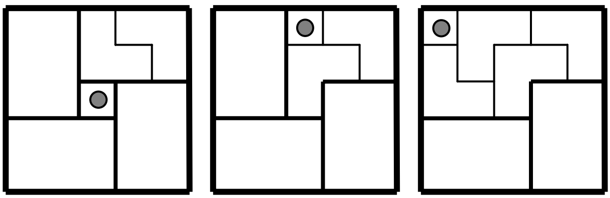

Figure 4

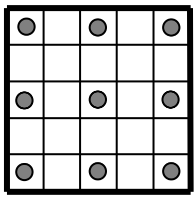

Figure 5

Remark. Such a proof is titled as the DEFICIENT 5×5 LEMMA (Ash & Golomb, 2003): "*If the square* (i, j) *is removed from the* 5×5 *rectangles where either i or j is even*, *then the resulting shape is not tileable.* We especially recommend the proof of the Deficient 5×5 Lemma to the casual reader." Let us explain that *deficient* "simply means there is one unit space" [8]. For a general case the following *Deficient rectangle theorem* is known. *An* $m \times n$ *deficient rectangle*, $2 \le m \le n$, *has a tiling*, *regardless of the position of the missing square*, *if and only if the following conditions hold*: (a) 3 divides $mn - 1$; (b) $m = 2 \Rightarrow n = 2$; (c) $m \neq 5$.

Unfortunately, this elegant proof has no generalizations for larger squares. For example, 7 × 7 is tileable regardless of the position of the missing square. Due to symmetry it is sufficient to examine 10 cells. One more efficient method is known, see Figure 6.

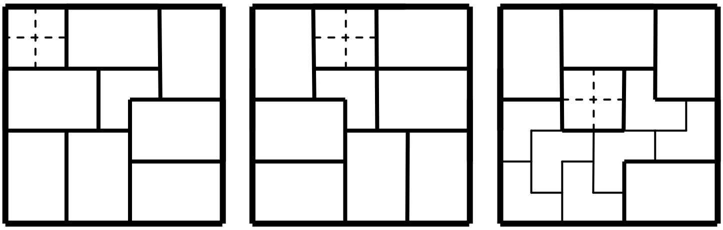

Figure 6

Further, a set of problems is offered mainly for the square 9 × 9 varying from easy to unsolved. In the following formulations instead of V-tromino we use tromino.

1. Find a *solution*, i. e. fill the tray 9 × 9 by trominoes.
2. Assemble 9 × 9 from *n* different V-rectangles if *n* = 2, 3, and 4.
3. Form the smallest V-square.
4. Form 9 × 9 with seven V-rectangles. Determine the minimum and maximum number of rectangles appearing in solutions of 9 × 9.
5. Cover 42 squares of the square 9 × 9 by 14 trominoes so that it is impossible to put the fifteenth tromino in the remaining region.
6. Find a solution having only one point where 4 trominoes meet.
7. Place 7 blue, 7 green, and 7 red trominoes in the square 8×8 so that there is only one contact point. Solve this task when a unique contact point is of type 2-2. Place 16 trominoes in the square 7×7 so that all three colours are separated.
8. Fill 9 × 9 so that two colours are separated.
9. Fill the 9 × 9 so that there are exactly *k* contact points. Determine all the values of the number *k* of contact points.
10. Divide the square 9 × 9 into three 10-gons assembled by trominoes.

P1. What is the minimum number c of the *crosses* for the square $3n \times 3n$? The dividing lines in the form "+" arise when tiling has a point where 4 trominoes meet, see Figures 2-3. What is the minimum number of contact points for the square $3n \times 3n$? The solution of the first problem does not always allow the desirable colouring. Find a solution of 9×9 having only one cross. Solve the m a i n t a s k of the *Three Colour Problem* for the tray 9×9. Try to find the solution with three contact points without assistance, if you fail see Figure 8. What is the range of number of contact points for the square $3n \times 3n$?

P2. Divide the square 9×9 into three V-polygons. Let a, b and c be the numbers of their edges. Determine the range of the sum $a + b + c$. Do the same if: 1) $a = b = c$; 2) all polygons are symmetric.

For more challenges contact the author.

Solutions or short comments

4. The minimum and maximum number of the rectangles 2×3 appearing in solutions of the square 9×9 is 4 and 11 respectively.

5. One can come across an easier task regarding the minimum number of V-trominoes for the square 8×8 in mathematical competitions. The idea is to divide the square 8×8 in 16 small squares 2×2. At least two unite squares of each square 2×2 must be covered by V-trominoes. Thus, it is necessary to have at least 11 trominoes to cover 32 unite squares. Now it remains to show (what is rather an easy task) that it is sufficient to have 11 trominoes. The solution for the square 9×9 is shown in Figure 7. Prove that 14 is the minimum number of trominoes. Solve the problem for the squares 11×11, 13×13, 15×15.

6. See **P1**.

7. The problem has been solved in (Klimova, 2004). The solution for the square 7×7 with a full separation of colours is shown in Figure 10.

P1. The solution with one cross is shown in Figure 8. This solution does not allow the colouring having only one contact point. The solution with the minimum number of contact points is shown in Figure 9. The maximum number of contact points for this square is 14.

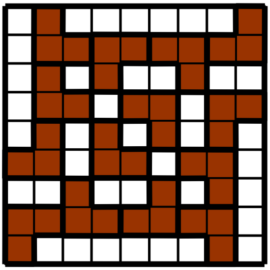

Figure 7

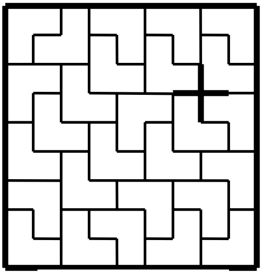

Figure 8

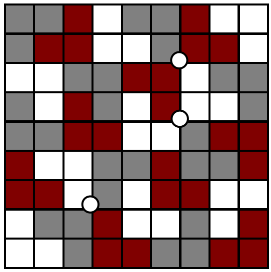

Figure 9

P2. This problem has been offered in mathematical camps and extracurricular lessons, and pupils welcome it with great enthusiasm. It can be proved that min($a + b + c$) = 20, see Figure 11, and 24 if $a = b = c$.

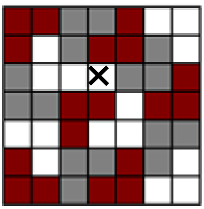

Figure 10

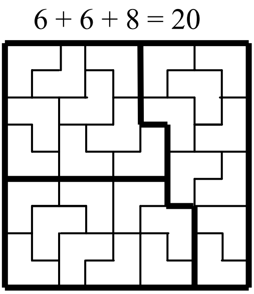

Figure 11

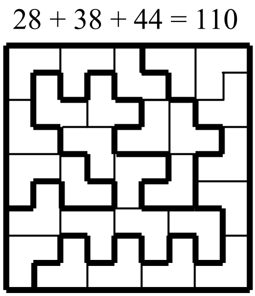

Figure 12

8 + 8 + 8

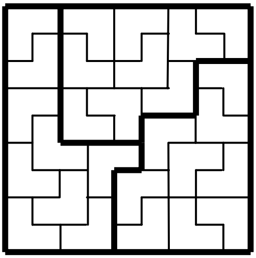

Figure 13

30 + 30 + 30

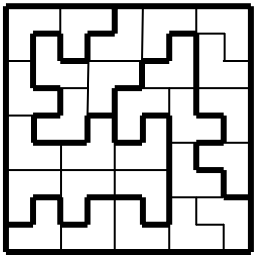

Figure 14

The problem of the maximum value is unsolved. At present the best solution having value 110 is shown in Figure 12. If $a = b = c$ then solutions for 2k-gones with all integer k from the interval [4, 15] are obtained in (Pure, 2009). The solution in Figure 14 with three 30-gones is taken from the bachelor's paper by E. Pūre (2009).

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PRACTICAL WORK ON SERIES: EDUCATIONAL ASPECTS

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Abstract

Examples of series of a different kind and mathematical level are considered. Attention is paid to didactic, historical, and non-trivial examples.

Introduction

There is no doubt that a great role in teaching of mathematics is played by examples, especially in practical or laboratory lessons. Diversity of opinions and questions may arise when we must choose concrete exercises, examples, problems. For example: Aren't these exercises too simple and, perhaps, too boring, or vice versa, aren't these examples too difficult? Should we offer only the well-known standard examples? What is the optimum amount and the level of exercises and problems that should be given to students as individual work? How to arouse interest and motivate students? How widely and deeply should we look at various characteristics for the exploration of convergence of series? A selection of exercises, enlightening examples or counterexamples as well as the way of presentation are influenced by many factors: textbooks, colleagues, experience, the background and the level of students' knowledge, etc. Different kinds of exercises on number and function series can be found, e.g., in (Кудрявцев et al, 1986; Кузнецов, 1983; Ляшко et al, 1977). The book (Кузнецов, 1983) is also available on the Internet and examples of solutions as well. Some solutions in (Кузнецов, 1983) are fallacious, and from time to time our students use this teaching aid non-critically. We will mention some mistakes made by students most frequently while solving basic exercises on series. Preparing the paper we looked through the paper of

E.Pehkonen (2009) and came across the cognition that we used in work with students: *The direction to emphasize problem-solving and self-initiativeness seems to be a correct one. New information should not be given in a "ready form", lecturer should led students via self-initiative and critical thinking.*

How to begin? How to arouse interest and motivate students?

It is well-known that historical examples are didactical and useful in order to improve teaching. However, they are not widely used in practice, e.g. because of shortage of time. We usually begin with the infinite series

$$S = 1 - 1 + 1 - 1 + \cdots,$$

and with the question: what is the sum of this series? After receiving students' answers (usually those ones: 0, 1, uncertainty, cannot be said) it is worthwhile introducing them to a short historical insight or at least references. In 1703, Guido Grandi (1671-1742) noticed that, on the one hand,

$$S = (1-1) + (1-1) + \cdots = 0 + 0 + \cdots = 0.$$

And, on the other hand,

$$S = 1 + (-1+1) + (-1+1) + \cdots = 1 + 0 + 0 + \cdots = 1.$$

It is easy to obtain also the value ½.

$$S = 1 - (1 - 1 + 1 - 1 + 1 \cdots) = 1 - S \Rightarrow 2S = 1 \Rightarrow S = \frac{1}{2}.$$

Is it the way how God could make something from nothing?

Remark. Bagni (2005) gives such *statistics* of students' answers regarding Grandi's series: 0 (29 %), 1 (4 %), 0 or 1 (20 %), ½ (5 %), infinite (2 %), the result does not exist (6 %), no answer (34 %). The sum of the alternating series was considered ½ by several outstanding mathematicians (Leibniz, Grandi, Euler, Fourier).

Two *heavier* ”proofs” that S is ½.

A geometric series (or Taylor series) yields:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \quad x = 1 \Rightarrow S = \frac{1}{2}.$$

At last *modern proof* is to be found in (*The College* …,2007):

$$\frac{1}{2} = \int_0^\infty e^{-2t} dt = \int_0^\infty e^{-t} e^{-t} dt = \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^n t^n}{n!} e^{-t} dt =$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^\infty t^n e^{-t} dt = = \sum_{n=0}^\infty \frac{(-1)^n}{n!} n! = \sum_{n=0}^\infty (-1)^n.$$

This historical and didactic example motivates the necessity of strong definition of the *sum* of series. Clearly, at the beginning, intuition played a more important role than rigor. After Leibniz's death, it took about 250 years to set Calculus on a reasonably solid foundation. What now seems simple and transparent for our students of mathematics (due to the clear definitions), used to be difficult and confusing even for the best mathematicians of the early days.

One of the older example of series is the famous paradox of Zeno of Elea (490-430 BC), named as Achilles and Tortoise. Extensive literature in question is found, e.g. (Grünbaum, 1968, Günther, 1997). *Achilles starts from* A, *the Tortoise at the same time from* B. (see Figure 1)…*This famous philosopher argued that while Achilles covers the distance* AB, *the Tortoise reaches point* C. *When Achilles arrives at* C, *the animal must have reached* D,… *and so ad infinitum*. *It follows, so Zeno concludes, that Achilles can never overtake the Tortoise*.

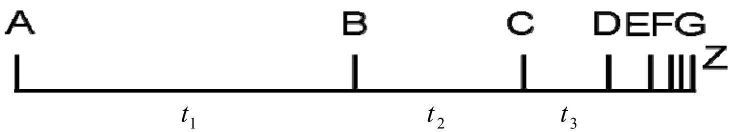

Figure 1

The confusion with Zeno's paradox is that Zeno (as well as many students in the introduction of the subject on series) was uncomfortable with adding infinitely many numbers. Let velocities of Achilles and the Tortoise be v_A and v_T respectively, and $v_A > v_T$. It would be a good exercise for students to calculate the sum of times $t_1 + t_2 + \cdots + t_n + \cdots$ which characterizes the time needed for Achilles to catch the Tortoise.

A *recurring decimal* is one of the most familiar infinite series. For example, the statement that $\frac{1}{3} = 0.333\cdots = 0.(3)$ really means that $\frac{1}{3}$ is the sum of infinity number of terms:

$$0.3 + 0.03 + 0.003 + \cdots = \sum_{n=1}^{\infty} \frac{3}{10^n}.$$

Geometric (visual) representations of series also help to present the material. The Mathematical Association of America has issued the envelope with the following formula and its geometrical representation by means of a sequence of triangles, see Figure 2. One fourth is the area of the largest white square provided that the initial triangle is of the unit area.

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1}{3}$$

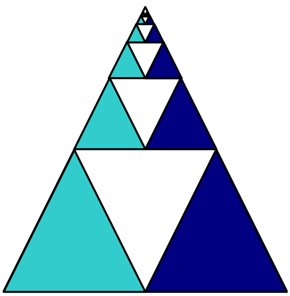

Figure 2

$$1 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

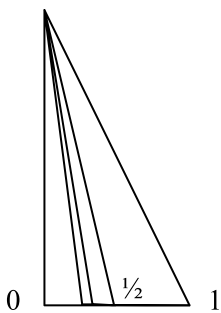

Figure 3

One can prove identity in Figure 3 also algebraically using the formula for convergent telescoping series:

$$\sum_{n=1}^{\infty}(u_n - u_{n+1}) = u_1 - u_2 + u_2 - u_3 + u_3 - u_4 + \cdots = u_1.$$

Taking $u_n = \frac{1}{n}$ in this formula we obtain $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$. Hence $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the Comparison Test. A lot of geometric representations is known for finite sums.

It seems that many textbooks and teaching aids offer only standard and boring exercises on series. If we work only with standard exercises, we train the "gentlemen" rather than provide a solid background in mathematics. To vary and improve teaching we recommend using **Yes or No Test**. During 10 to 15 minutes students are required to answer whether the given statements (about series) are true or false (explanations are not necessary). These tests contain not only standard questions but also a few ones being more difficult. To illustrate we have provided one sample test. This test is used after the students have listened to the appropriate theoretical lecture on series.

Example of Yes or No Test

① $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = q \in R:$ $a)$ $q > 1 \overset{?}{\Rightarrow} \sum a_n \notin R;$ $b)$ $q < 1 \overset{?}{\Rightarrow} \sum a_n \in R$ ◯ ◯

② $\frac{a_{n+1}}{a_n} \uparrow 1 \overset{?}{\Rightarrow} \sum a_n \in R$ ◯

③ $\frac{a_{n+1}}{a_n} \downarrow 1 \overset{?}{\Rightarrow} \sum a_n \notin R$ ◯

④ $\sum |a_n| \in R \overset{?}{\Leftrightarrow} \sum a_n^2 \in R$ ◯

⑤ $a_n > 0,\ na_n \to 0,\ n \to \infty \overset{?}{\Rightarrow} \sum a_n \in R$ ◯

Students have to write “+” or “–” in the circles. After receiving answers the lecturer gives the short solutions, comments. If more than a half of the number of student’s answers is right he/she has passed the test, otherwise a student must fill the test at home with full solutions. Right answers to these questions: 1. a) Yes; b) No. 2. No. 3. Yes. 4. No. 5. No. *Hint*. Examine the series with the general term: $a_n = (-1)^n$; n^{-1}; $(n\ln n)^{-1}$.

The next five examples would be more difficult: 1. Let (C) and (D) be the following two statements about series with positive terms: (C) Series can be proved convergent by the nth Root Test; (D) Series can be proved convergent by the Ratio Test. (In Russian literature these tests are known as *Test of Cauchey* and *D’Alembert* respectively.) Is it $(C) \Leftrightarrow (D)$? As a rule, students’ answer is *Yes*, however, the right answer is *No*.

2. $\lim\limits_{n\to\infty} \dfrac{a_{2n+2}}{a_{2n+1}} = q < 1 \overset{?}{\Rightarrow} \sum\limits_{n=1}^{\infty} a_n \in R$;

3. $a_n > 0, \sum a_n \in R \overset{?}{\Rightarrow} na_n \to 0, n \to \infty$;

4. $\sum\limits_{n=1}^{\infty} a_n \in R \overset{?}{\Rightarrow} \sum\limits_{n=1}^{\infty} a_n^3 \in R$; (See Кудрявцев Л. Д. et. al., 1986 p. 311)

5. $S_n := \sum\limits_{k=1}^{n} a_k, |S_n| \underset{,n\to\infty}{\to} S_0 \in R \overset{?}{\Rightarrow} \sum\limits_{k=1}^{\infty} a_k \in R.$

We recommend paying the attention to **mistakes or irrational solutions** one can find in textbooks or teaching aids. Let us consider an example from the book (Кузнецов, 1983) provided with the following solution at [http://reshebnik.ru/solutions/6/12/]. *Find the region of convergence of the functional series*

$$\sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^n \frac{1}{\sqrt{n}} x^{2n} \cos(x - \pi n) = \sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^n \frac{(-1)^n}{\sqrt{n}} x^{2n} \cos x.$$

Let us use the Ratio Test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{5}{3}\right)^{n+1}\frac{(-1)^{n+1}}{\sqrt{n+1}}x^{2n+2}\cos x}{\left(\frac{5}{3}\right)^{n}\frac{(-1)^{n}}{\sqrt{n}}x^{2n}\cos x}\right| = \frac{5}{3}x^2 < 1 \Rightarrow x \in \left(-\sqrt{\frac{3}{5}};\sqrt{\frac{3}{5}}\right).$$

The convergence at the endpoints have been examined by the Alternating Series Test (Leibniz's Test) and the following answer has been obtained: $x \in \left[-\sqrt{\frac{3}{5}};\sqrt{\frac{3}{5}}\right]$. The answer is wrong! We hope that the reader will find easily the mistake in the given solution.

Find whether the series $\sum_{n=1}^{\infty}\ln\frac{n^2+5}{n^2+4}$ *is convergent*? Some clever students solved this exercise by Integral Test. A few lines of one long and cumbersome solution:

$$\int_1^{\infty}\ln\frac{x^2+5}{x^2+4}dx = \int_1^{\infty}(\ln(x^2+5)-\ln(x^2+4))dx = \ldots =$$

$$= x\ln\frac{x^2+5}{x^2+4} + 2\sqrt{5}\,arctg\frac{x}{\sqrt{5}} - 4\,arctg\frac{x}{2}\Bigg|_1^{\infty} = \ldots$$

$$= 0 + 2\sqrt{5}\cdot\frac{\pi}{2} - 4\frac{\pi}{2} - \ln\frac{6}{5} - 2\sqrt{5}\,arctg\frac{1}{\sqrt{5}} + 4\,arctg\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}\ln\frac{n^2+5}{n^2+4} \in R.$$

The efficient solution. As the series $\sum\frac{1}{n^2}$ converges, and $u_n = \ln\left(1+\frac{1}{n^2+4}\right) \sim \frac{1}{n^2+4} \sim \frac{1}{n^2}$, $n\to\infty$, then $\sum u_n$ also converges by the Comparison Test.

Example. Prove or disprove that both series $\sum n(a_n - a_{n+1})$ and $\sum a_n$ converge or diverge simultaneously. This is the solution found by one student:

$$\text{“}\sum_{n=1}^{\infty} n(a_n - a_{n+1}) = a_1 - a_2 + 2a_2 - 2a_3 + 3a_3 - 3a_4 \cdots = \sum_{n=1}^{\infty} a_n .$$

Therefore both series converge or diverge simultaneously.”

Find a mistake in this very plausible proof. *Hint*. Let $a_n = 1$. Then $\sum n(a_n - a_{n+1})$ converges, but $\sum a_n$ diverges.

We conclude the paper with the conclusion: None of methodology is able to diminish significantly the difference between the level of mathematics knowledge of the best and worst students.

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The College Mathematical Journal, Vol. 38, No. 1, January 2007.

USE OF ICT IN MATHEMATICS TEACHING

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Abstract

Information and communication technologies (ICT) make learning process more creative and suitable for students. Internet increases "borders of time". That gives an opportunity for each student to learn in suitable speed and personal skill level.

ICT possibilities let each student individualize their learning. Hereto the teachers continue to play important role in making and structuring the learning process. Implementation of new technologies brings radical changes in education system: there were teacher in the centre before, now – a student. Teachers are more like advisers who try to help students to perform their own personal made way in different "brush-wood" of network information.

What would you see in a technology-integrated classroom

What might you see in a classroom walk-through? To see or experience some proofs of what is indicated above would require discussion with the teacher and students, a longer observation period or analysis of planning. The list below indicates some of the things that should be readily apparent when you enter a technology-integrated classroom:

- Technology in the hands of students and the teacher,
- Collaboration with others in the room, the building, or globally;
- Students creating;
- Creations for larger and authentic audiences;
- Students doing different things in the same room at the same time;
- Decentralized classroom layouts with technology dispersed throughout the room;
- Access to technology in the room is student controlled;

- Teacher in small group or one-to-one discussions;
- Teacher modelling the use of technology to create, collaborate, locate, evaluate, and communicate information;
- Teacher modelling of an instruction in discourse;
- Content rich explorations and activity conducted by the students;
- Content rich discussions among the students.

Everyone that has ever used ICT effectively in the classroom to support pupils' learning will know the power of technology to open young people's minds to the fascination and inspiration of mathematics.

One of the subjects, where the use of ICT is being done and studied at the same time is Mathematics. Since mathematics is abstract in nature, teachers are constantly looking for ways or tools to help their pupils understand the underlying concepts of the lessons. One of these tools is ICT. According to Cunska & Malkalne (2008), numeracy becomes better when teachers use and allow their students to use resources including ICT in order to model mathematical ideas and methods. In addition, ICT is seen as a tool that will be able to help students with problem-solving.

Majority of the student population has always perceived mathematics as a difficult subject. This is the reason why many students also find it difficult to learn the ideas behind the subject, they make themselves believe that it is not doable in their case. However, the use of ICT is promising to change the perspective of both- students and teachers towards learning and teaching mathematics. Therefore, there is a need to investigate the effects of ICT in learning and teaching mathematics in order to help students and teachers to improve their performances in the field of mathematics.

Latvian government, local authorities, companies and different projects had made a huge investment in ICT in schools over the past few years to provide computer technology, software and training of teachers.

Much less has been invested to develop ICT-related learning content, learning materials and methodology.

80% of Latvian math teachers have a computer with Internet connection at home, but majority of them keep teaching in a traditional way.

The five most popular application packages used by mathematics teachers are word processing packages (71.1%), spreadsheets (51.2%), search engines (44.1%), presentation software (36.9%) and drill and practice software (24.3%). However, it must be noted that the packages that have not been positively considered by the respondents may not be necessarily useless. Mathematics teachers need more time to learn to use them – programs like specific Java applets, Flash presentations, graphical applications and simulation programs have great potential for the teaching of mathematics as they encourage explorations and higher level thinking.

The most creative teachers develop learning materials themselves and insert them into educational online environments:
- Latvian education information system project web page www.liis.lv;
- web page supported by Microsoft – Latvia www.skolotajs.lv;
- virtual learning environment of Jelgava regional adult education centre http://ikt.jrpic.lv;
- Auce High school homepage http://fs-it.blogspot.com;
- web page supported by Miksike and Smiltene gymnasium lv.lefo.net;
- learning objects library of Riga Technical University http://213.175.92.77/course/.

EU project for development of the content of science and math in secondary level was carried out in Latvia from 2005 till 2008. The project goal was to ensure an opportunity for students to use different learning methods and contemporary technologies.

The results of this project were:
- advanced learning content in secondary education level in math;
- 15 supported material sets had developed, published and enforced;
- chance for 50 schools to become a regional methodological centres;

- opportunity for 2950 teachers to increase their qualification in in-service training programs.

Within the project following outcomes were made:
- useful visual materials in electronic form for learning of every theme: pictures, interactive animations and presentations;
- interactive course for autonomous learning of students. Each theme of the course includes short theoretical material, interactive visual material, task examples with solving samples, self- solving tasks with answers, tests for self -control.

Since December 15, 2008 EU project is carried out to develop the content in science and math for elementary school (grades 7.-9.).

The role of technologies in school development

The experience of Smiltene Gymnasium in using technologies and collaboration in education process started in 1993 when the first computer class with 15 computers was set up. During the following 16 years students and teachers have become the participants in international ICT projects, creators of the content, ICT experts and lecturers of education courses. We consider the effective usage of technologies as the main transforming element that can create a new model of school, preparing students for the information society age. To make it possible it is not enough to have schools provided with technology – educational content, critical thinking, skills and acquirements to apply ICT are also needed.

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ADVANCED EDUCATION AT SCHOOL LEVEL: ACHIEVEMENTS AND TRENDS

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Abstract

The role of mathematical contests and other activities in advanced math education in the Latvia has been examined. The possibilities for junior students and high school students to participate in the correspondence contests and supporting activities for these contests organized by "A.Liepa's Correspondence Mathematics School of University of Latvia" have been described.

The mathematics curriculum in Latvia has changed considerably during the last 30 years (see Freija, 2007). Constant decrease in the quality and number of exact disciplines taught at school can have an adverse effect on young people by decreasing their ability of independent process analysis, which in turn can lead to an inferior role in society. The European project "Math and Sciences" was carried out in Latvia in the period from 2005-2008. As a result of this project some changes are evident in the approach to teaching mathematics in Latvia; however, but they are still insufficient. It will take a long time to make considerable and sufficient changes in the curriculum of mathematics.

The system of extra-curricular activities, a considerably less formal and conservative approach, is playing an increasing role in advanced education. Thus mathematical contests and other activities in advanced math education have become an essential part of school education in Latvia. Such activities as correspondence courses, correspondence contests, olympiads, students' research clubs etc., run by enthusiastic teachers and university students, greatly complement the official curriculum. The material covered by these activities is much broader; the scientific standards are higher; and the possibilities to compare one's achievements with those of counterparts in the whole of Latvia are broader.

For almost 40 years a scientific/methodical laboratory presently bearing the name "A.Liepa's Correspondence Mathematics School of University of Latvia" (CMS) has been functioning at the University of Latvia. It was founded in 1969 with the goal of preparing high school students for the entrance examination of the Faculty of Physics and Mathematics at the University of Latvia. As the entrance examinations were very hard at those days, there was a need for extra studies to prepare for them. CMS mainly practiced extramural math teaching – there was correspondence between students and CMS on the university's entrance examination level.

For some years there are no more university entrance exams, so in order to become a student you have to pass high school graduation exams very well. The abovementioned changes in the system of education in Latvia during the last 30 years have had a great effect on these exams. As there many essential topics are deleted from the school curriculum, the exams are getting easier and the level of students' knowledge is getting lower. CMS has followed all the changes in the school curriculum for all these years and has changed as well, but in a quite different way. CMS has noticed that there are many topics missing from the school program which are essential for the entire comprehension of math . So CMS is now trying to provide the possibility to study these topics, by organizing correspondence contests and supporting activities for these contests.

Extramural studies for high school students (ESH)

On September at the beginning of the school year an application to the running ESH year is announced on the Internet website of CMS http://nms.lu.lv. Any 9^{th} to 12^{th} grade Latvian school student can join the ESH. Through the school year there are 4 or 5 ESH lessons so 4 or 5 booklets are prepared and sent to students who have decided to participate. Booklets contain elements of theory, examples and problems for independent solution. The students have one month to solve them, then they have to send their solutions to CMS. The supervisor of this contest collects the sent-in solutions, then search for the people that can check these solutions. Usually the checking is done by CMS collaborates and students of the University of Latvia. The supervisor also gives some hints to these people. When the sent-in solutions are checked and commentaries are written on the paper, the supervisor collects the checked and commented solutions and checks them once more. After that the next booklet is sent out. It contains not only theory, examples and problems, but also solutions to the previous booklet's problems. The checked papers are returned to the participants as well.

Over the years ESH lesson issues have changed according to the necessity. As CMS has been dealing with student education for many years, it has great experience of the problems the students face. So ESH can choose the topics that are fundamental in mathematics but still are difficult for the students.

Analyzing research papers in mathematics and computer science, we have concluded that following general methods are of particular importance:

- Method of mathematical induction;
- Mean value method (generalized Pigeonhole Principle)
- Invariants
- Extremal element method (the set is analyzed by investigating its extremal elements)
- Interpretation method (the problem is "translated" into appropriate language).

Correspondence contests for junior students

CMS also organizes the "Young Mathematicians' Contest" (YMC) for 4-7th graders. Some younger students take part in it as well. YMC is a contest of middle difficulty, where full solutions are required. This contest is a good way to introduce students to problems that are similar to problems in math olympiads.

The strongest participants of YMC usually also enter the other contest - "Professor's Littledigit Club" (PLC). It is a high difficulty level contest for 4-9th graders. There are problems up to unsolved topics for research so this contest is for those students who have done a lot of work with math problems before and want to achieve considerable results in this field.

A lot of teachers, students and parents have expressed interest in and necessity for similar contests for primary school pupils. So contest "So much or… How much?" for 4th Grade students was established. It is an international correspondence enterprise. In this contest there is one preparatory round, two regular rounds and one final round organized by CMS. Only the winners of those four rounds are invited to the final round that is a joint Latvian-Lithuanian competition.

Supporting activities

CMS organizes a "Little Mathematics and Informatics University". Every month there is one Saturday when lecturers of the University of Latvia give lessons to high school students. There is one math lesson, one informatics lesson and one practical lesson on computers. These lessons take part at the University of Latvia in Riga, but students from many regions of Latvia also participate in them.

There are also monthly sessions for students and teachers in some back-country regions that are organized during school year. The workers of CMS go to these schools once a month. Students and teachers really appreciate the chance to obtain math knowledge not far from their home.

When the school year has ended mathematical summer camps are organized for students in some regions. CMS also takes part in this event.Beside the present-time activities and competitions, a lot of teaching aids are prepared and published for advanced math education. There are methodical materials on some topics and volumes of problems with extended solutions of Latvian math Olympiads and other contests. Teaching aids are published both in a "hard copy" version and on the Internet. A lot of books are written and published within the Latvian – Icelandic project "LAIMA". Some of them are translated into English as well.

At the University of Latvia a system of courses for future teachers of mathematics is developed within the bachelor's and master's programs. In these courses a great attention is paid to the methods of work with gifted students at school.

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SOME IDEAS ON TEACHING MATHEMATICS IN LATVIA IN THE 1920S–30S AND NOWADAYS

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Abstract

In the 1920s-1930s textbooks on methods of teaching mathematics were published in Latvia as well other pedagogical literature on mathematical studies. Quite a number of recommendations have not lost their topicality. The most outstanding Latvian specialist of the 1920s-30s in methods of teaching mathematics was Longīns Ausējs (1885 - ?).

In the 1920s and 1930s five teacher training establishments functioned in Latvia, where methods of teaching mathematics were taught. These establishments were Rīga, Jelgava, Daugavpils and Rēzekne Teacher Training Institutes as well as University of Latvia. Graduates from the six-year primary school were admitted and it took 5 years (after 1929 – 6 years) to train them as primary school teachers. Also the future secondary school mathematics teachers studied at the University. So, there was a need for course-books in methods of teaching mathematics.

The first of these books had two authors – Russian specialist in methods Teodors Erns (1863–1926) and Oskars Priedītis, a teacher of mathematics and lecturer in methods in Jelgava Teacher Training Institute (1893–1941). After T.Erns' death, until the year 1935, O.Priedītis was the only author of such books in Latvia. Separate parts of his "Methods of Teaching Mathematics" are devoted to teaching of geometry and algebra. Up to the present nothing comparable (in terms of firm focusing on the specific features of algebra and geometry) has been published in Latvian.

O.Priedītis' appeal that in the process of learning mathematics lots of attention should be devoted to discoveries and experiments sounds very relevant even nowadays. He wrote: “Geometrical notions and truths are found out and made completely clear by *experiment, geometrical intuition and logical thinking*” (Priedītis, 1932, III, p.8). When discussing teaching proofs in geometry, O.Priedītis urged not to hurry: “It is possible to teach to think in an abstract way only at a slow pace and following a strict gradual sequence. Otherwise learners will mechanically learn the given proofs by heart and “will answer” them quickly and fluently but nothing is reached that way”. Even on the secondary level a lot of themes in geometry, for example, the first theorems in stereometry should be approached in a very concrete way”.

The author of the second textbook was lecturer in methods of teaching mathematics at Rīga Teacher Training Institute and Docent of the University of Latvia**,** the outstanding Latvian pedagogue and public figure Longīns Ausējs (1885–?). In the introductory part of his “Methods of Teaching Arithmetic” he wrote: “I have used the sheer abundance of sources on methods of teaching mathematics in the German, Russian, English, French, Danish, Swedish and Czech languages, carefully screened it and supplemented it with my own observations” (Ausējs, 1935a, p.3). The content of this book proves that a really wide spectrum of foreign literature sources on methods has been used.

Nowadays mathematical education shares the aims defined by L.Ausējs in his time. He has written: “Life requires that an educated person should possess sufficient ability to make judgements, act and perfect oneself” (Ausējs, 1935a, p.57). In other words, already in the 1930s the aims of mathematical education were: 1) to enhance development of logical thinking; 2) to give knowledge needed for life; 3) to develop self-study skills.

In 1938 Longīns Ausējs’ article “*The Desirable and the Attainable at School*” was recognised to be the best in Latvia; it was even published as a separate booklet. As we can see from the title, this work bears a general pedagogical and even philosophical character, which entirely refers also to learning mathematics. Concerning the eternal contradiction between the desirable and the really attainable in the area of students' knowledge, skills and abilities L.Ausējs suggests to tackle it with the help of purposeful selection and structuring of the teaching/learning material: *"The only correct approach – never strive to achieve the unachievable, i.e., either decrease the amount of the study material, or at least split it into clearly defined parts"* (Ausējs, 1938, p.22). L.Ausējs' suggestions how to select the study material are described in figure 1. His ideas about the need to define the core in the teaching/learning material and structure it, were further developed by L.Ausējs' successor Jānis Mencis (born. 1914).

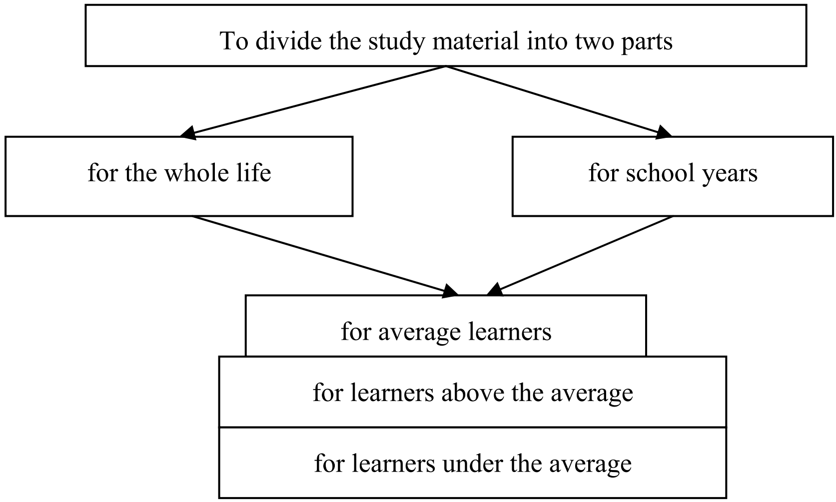

Figure 1. How to select the study material

Important even for today's teachers is L.Ausējs' recommendation: "May the first and the major, never to be forgotten, fundamental rule in this field be: "Never go into the classroom unprepared!" (Ausējs, 1935,

p. 69). In order to prepare for the lesson the teacher should consider four things: 1) how to split the study material into elements, and to arrange them into ascending order of complexity, 2) how to achieve the learner's understanding of the study material and arouse their interest; 3) how to follow the regularities of psychology and the principle *Festina Lente* – hurry up but slowly! 4) how to make the mathematical problems and practising as interesting as possible.

In these four activities L.Ausējs sees the very essence and creative nature of the mathematics teacher's work. He stresses: "And here is the field of feasible searches and improvements, in which the teacher can think and work without being driven into boredom, never being able to achieve perfection! Outside the four mentioned cornerstones there is always room for disappointment and feeling of going astray, whereas in their confined space the searches will bring gratification each time" (Ausējs, 1935a, p.64).

In his geometry book Longīns Ausējs' has presented recommendations for learners (and, most likely, also for mathematics teachers) how to develop their general problem- solving skills (Ausējs, 1932, pp. 45-46). If we compare these recommendations with the suggestions in Polya's (1945) classical work *How to Solve It* we can see strong similarities. Both authors define 4 stages in solving a problem (Polya's definitions have been used):

To understand the problem;
To devise a plan;
To carry out the plan;
To look back.

It is generally recognised that learning mathematics facilitates the learners' intellectual honesty, which can be linked with ethical behaviour in general. The views on life philosophy and ethical views voiced by L. Ausējs can be linked with intellectual honesty. One of his books bears a title "Roads to Happiness" (Ceļi uz laimi"), and presents the following "algorithm" of happiness: 1) building up one's own potentials; 2) emphasising the positive and overcoming the negative; 3) enhancing others' well-fare and happiness; 4) self-development; 5) serving some higher goals (Ausējs, 1937). In an another book by L.Ausējs the following "harsh life rule" was defined and defended: "there is nothing lost in life, but everything pays off, i.e. he who gives gets it back, similar to a situation when a stone thrown into a quiet pond creates waves spreading in all directions, they reach the shore, beat on it and then retreat to the starting point; however, sometimes the retreat is so slow that the wave reaches only our children or grandchildren" (Ausējs, 1935b, p.6).

In conclusion I would like to mention some data from O.Priedītis' and L.Ausējs' biographies.

Figure 2. Oskars Priedītis

Figure 3. Longīns Ausējs

Oskars Priedītis graduated from Valmiera Teachers' Seminary, but got the rights of a secondary school Mathematics teacher as an external student. In the period between 1922 and 1941 he was a teacher of mathematics and methods of teaching mathematics in Jelgava State Teacher Training Institute. Oskars Priedītis was a highly intellectual and respected educationalist, who was given the highest award of Latvia – Three-Star Order (Triju Zvaigžņu ordenis) for his conscientious and selfless work. The last two years of his life were darkened by a tragic mistake: in 1940 O.Priedītis was pleased to welcome Latvia's occupation; therefore he was promoted to the position of vice-principal or inspector of Jelgava Teacher Training Institute. However, soon he became aware of his terrible blunder – that a bloody dictatorship had seized the power in Latvia. According to some information, O.Priedītis committed a suicide. However, further work with the archives is required concerning the matter whether there might have been someone who "helped" him pass away in 1941.

Longīns Ausējs graduated from the University of St. Petersburg Faculty of Mathematics, since the year 1910 he worked as a teacher, since 1911 as the school principal. He wrote the first Latvian algebra course-book and was the author of school arithmetic, algebra and geometry course-books throughout a long period of time – from 1918 to 1939. L.Ausējs taught methods of teaching mathematics at Rīga Teacher Training Institute and University of Latvia (1921–1940). He is well-known as an outstanding Latvian educationalist, philosopher and public figure: L.Ausējs was Mayor of the Town of Cēsis in 1918, member of Rīga City Council in 1925, Saeima deputy and Chairperson of the Education Commission (192-1931), founder of the Secondary School Teachers' Association in Latvia. L.Ausējs was arrested by Soviet occupation authorities in 1940, deported to Russia and most likely perished in Siberia.

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TEACHING BETTER MATHEMATICS-
A DEVELOPMENTAL RESEARCH STUDY

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Abstract

In the paper two linked developmental research projects are illustrated and discussed. Inquiry communities are built in workshops where teachers and didacticians work together in order to develop mathematics teaching. Improved mathematics learning for pupils is the intended outcome.

Introduction

The research project Teaching Better Mathematics (TBM) is a continuation study building on two earlier projects called Learning Communities in Mathematics (LCM) and Information and Communication Technology in Mathematics Learning (ICTML). In the projects didacticians are working together with mathematics teachers in schools, trying to develop mathematics teaching and improve pupils' learning in mathematics. Schools and preschools are separately funded in a parallel, linked project – Learning Better Mathematics.

Participants in the projects TBM and LBM

The TBM-project includes eight didacticians and two doctoral students. Four preschools, six compulsory schools, and three upper secondary schools have volunteered to take part in the activities and are part of LBM. The TBM-project 2006-2009 is financed by the Norwegian Ministry of Education and Research.

The aim of LBM (Learning Better Mathematics)

LBM will support teachers in preschools and schools to develop and try out working formats, tools and methods, which will allow them to create a good learning environment, good subject development and social development, with the school subject mathematics as the focus. Additionally each school and preschool has set its own goals. Learning by inquiry is a dynamic and interactive process, which is used by all participants as both means to achieve the decided outcome and part of the goal. Inquiry is realised through the presence of six elements:

Questioning	Pupils (& teachers) are encouraged to ask questions
Investigation	Pupils investigate and collect information
Creation	In this way new knowledge is created
Discussion	Pupils discuss the new knowledge
Reflection	Pupils reflect on the new knowledge
Wondering	Discussion and reflections leads to wondering, which raises new questions

The didacticians justifications and theory for the research projects

The current status of mathematics learning in Norway

International studies such as TIMSS, PISA, and national studies such as KIM and the L97 evaluation (Lie, Kjærnsli & Brekke; Brekke, 1995; Alseth, Breiteig, & Brekke, 2003) indicate that there is potential to improve the performance in mathematics for Norwegian pupils. Didacticians consider teachers' competence in mathematics and ways to approach the teaching of mathematics important aspects to work with. Research and development work, which leads to sustainable changes in the education system, needs to be accepted as long term projects and are meant to wait for outcomes with some patience. Thus it is important for the didacticians to collaborate with teachers over a long period of time. A report from the two first projects is available (Jaworski, Fuglestad, Breiteig, Bjuland, Goodchild & Grevholm, 2007).

Communities of inquiry

The philosophy of teaching and learning mathematics in the projects is based on two fundamental theoretical constructs, *inquiry* and *community*. The work in the projects rests on the building of *inquiry communities* which means, "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (Wells, 1999, p. 122).

Methods of collaboration: Workshops as joint activity

The project creates opportunities for joint activity between teachers and didacticians. Part of the community building takes place in workshops at the university and in schools or preschools. Ideas of inquiry for different age levels are introduced and joint exploration takes place. The schools and preschools participating do so with commitment from the school leaders and at least three teachers from each school or preschool in the project. In following and documenting the work, focus has been on what inquiry means in schools and preschools, both in planning for mathematics teaching and in the classroom.

"Research is systematic inquiry made public" (Stenhouse, 1984). What does this mean for the didacticians? What is it we are inquiring about? How can we inquire in systematic ways? How do we make the outcome of our inquiries public?

What is the research in TBM about?

How do the didacticians and teachers go about researching what we are inquiring about? What methods do we use and why? We started with research questions about the development of mathematics teaching and how to make it sustainable. Teachers develop teaching sessions, which are documented and later analysed and researched by teachers and didacticians together (Hundeland, Erfjord, Grevholm, & Breiteig, 2007). We inquire into mathematics, into mathematics teaching and into mathematics learning.

The main aims and goals of the TBM project at UiA are to develop knowledge and practice in the teaching and learning of mathematics, so that pupils in schools have better learning experiences and achieve better conceptual understandings of mathematics. Pupils should demonstrate fluency in mathematics based on deep understanding. (See http://prosjekt.hia.no/tbm/).

Research questions in several areas

Didacticians and teachers ask about mathematics classrooms and about

a) Students, mathematics, curriculum, assessment
b) Mathematics teaching: Knowledge and approaches
c) Inquiry in mathematical problem-solving for students and teachers

Further we investigate information and communication technology in mathematics learning and teaching (ICT), communities of learning and inquiry within mathematics learning and teaching.

The developmental research cycle

We work with development and research together in a developmental research project, where the development carried out in classrooms gives opportunities for data collection to the research part. The analysis of data in the research part leads to new ideas for development that can be carried out in the schools together with the teachers. Development and research are closely linked and inspire each other. A workshop could be planned from a developmental perspective and when it is recorded it provides data for the fundamental research questions we ask. Outcomes from parts of research, such as analysis of recordings or pupils' work, give foundations for developmental parts in the workshops.

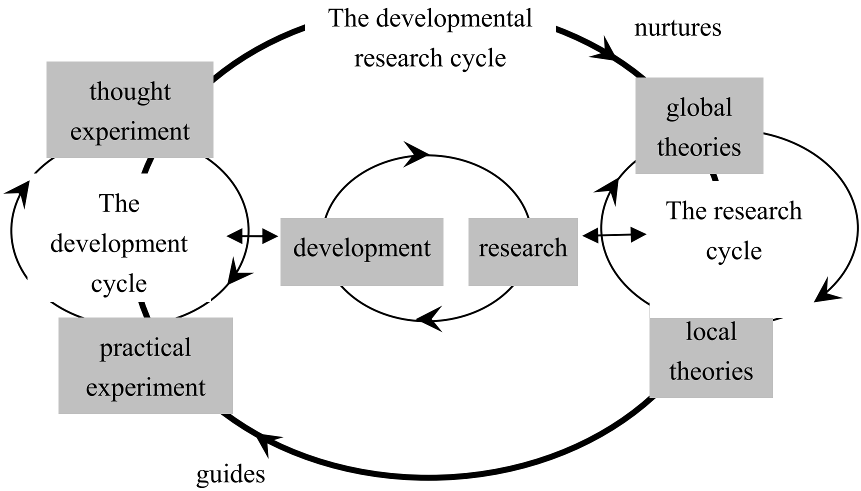


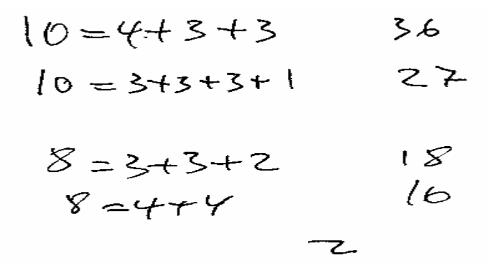
Figure: The developmental research cycle, (Goodchild, 2008. p. 208)

Who are collecting data in our projects?

Besides the eight didacticians at UiA and two doctoral students all preschool teachers in the projects (about 20 teachers), and all teachers in the projects (about 30 teachers) are observing and collecting data. All participants in the projects are seen as researchers, and are contributing with their special knowledge to both development and research as a result of the co-learning partnership and ownership of the project. A common database is created at UiA, where all data is stored in a systematic way and data can be used by any researcher in the projects. An events calendar linked to the date and time of each event leads to the files related to that event. Files can be audiorecordings, video-recordings, text-files, pictures, transcripts, data-reductions, or scanned material (pupils' work). Recorded events are analysed in the original language. One reason for the use of a common database is for the co-learning partnership and community of inquiry to be open for mutual support and collaboration. Further didacticians help each other to find relevant parts for answering one participant's questions, when analysing recordings and events. The learning community opens for democracy and equity. All can influence and take initiative to collect data and use it. All have a voice in the community and rights to use common material. Teachers in the projects carried out interventions and collected data in schools and preschools in many different ways. Their observations are often used as a basis for common discussions and presentations in later workshops at UiA. Teachers have written about, presented and documented their experiences (Grevholm, Billington, Skagestad, & Grostøl, 2008). Why do we see teachers as researchers? In the learning community of inquiry all are seen as researchers. Teachers' knowledge and experience are valued and important for the projects. Teachers' expertise is necessary for the development in schools and the partnership between teachers and didacticians rests on shared responsibility and ownership of work.

An example from inquiry in a workshop may illustrate the work:
In advance the didacticians have prepared an inquiry task to solve collaboratively. The task presented on the board: Write 10 as a sum. Multiply the terms of the sum. What is the biggest product you can get?

Try with other numbers than 10. What do you find?
Pair work in the audience is presented. After some suggestions a pattern becomes visible: Many terms as 3 or close to 3 is rewarding?



Now this hypothesis can be further investigated and refined. Teachers bring the task to their classroom. Pupils can work with this task and in doing so they partition numbers in many ways, add terms and multiply factors, which they have themselves chosen. In one upper secondary class the whole numbers were replaced by real numbers and interesting results came out. The open character of the task invites pupils to doing mathematics at a level of demand that suits them.

Concluding discussion

Evidence can be found and has been reported that teachers in schools and preschools are inspired by inquiring into and doing mathematics collaboratively in the workshops (Jaworski, Fuglestad, Breiteig, Bjuland, Goodchild & Grevholm, 2007). Teachers bring the new experiences to their classrooms and try out ideas with the pupils. Thus the teaching develops and teachers are reflecting more carefully about the learning opportunities they offer to the pupils. What is difficult for us to document (or measure?) is how this development of teaching influences pupils' learning. Other studies have reported that the engagement of the teacher is crucial for the learning outcome, which indicates that the teacher development initiated in the projects has consequences in classrooms. The expertise of the teachers and their judgement guide them in the decision about how to use inquiry for meaningful learning of mathematics.

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REPETITION IS THE MOTHER OF KNOWLEDGE

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Abstract

The role of repeated problems in mathematical contests in Latvia is considered. Also the positive and negative aspects of this praxis and possible solutions to minimize negative features are discussed. Some examples of repeated problems are offered.

The system of contests is the only part of school mathematical education in Latvia that has preserved the high standards achieved in the 80-ies/90-ies of the last century.

In collaboration with Ministry of Education and Science and other organizations, every year the A.Liepa Correspondence Mathematics School (CMS) organizes three rounds of State Mathematics Olympiad. In the first two rounds (Preparatory and Regional Olympiad) each pupil can attend. Best participants of Regional Olympiad are invited to the third (final) round of the State Olympiad. Also Open Mathematical Olympiad is organized. Unlike in State Olympiad, everyone can participate here.

Mathematical Olympiads differ a lot from traditional tests and even from exams in mathematics. Students must solve more profound, original and somewhat untraditional problems. Thus students must be not only well- aware of common definitions, theorems and methods from traditional school mathematics curricula, but must also invest their efforts and develop their creativity.

To solve these problems the students must prove what they say; quite long expositions must be composed and written in a clear and precise way, using grammatically and literarily correct language. Very diverse mathematical content and levels of difficulty appear regularly in olympiads, correspondence contests etc.

As all of us know the people are learning almost exceptionally from examples. Creative studies of last years' contest problems is one of the strongest tools towards the success this year, an effective way to acquire a lot of knowledge and skills usually staying beyond the official school curricula but containing deep ideas and approaches accessible even for Grades 5 and 6.

To stimulate such studies, the praxis of the so-called "repeated problems" is introduced into some Latvian math contests for approx. 15 years. Namely, each year one problem out of five is the same (or slightly changed) as some that was given in the same grade one or two years ago. Usually it is neither the easiest nor the hardest one. The changes are not principal, e.g., cities connected by roads instead of mutual friendships among children, some odd natural number instead of another one, etc.

Example 1.

35th Open Mathematical Olympiad, year 2008.

In one planet there are 2008 languages used. What is the minimal number of dictionaries necessary to translate from each language to any other? (Multilevel translation is allowed; each dictionary is used to translate only in one direction, e.g., from Latvian to Estonian, not vice versa.)

36th 35th Open Mathematical Olympiad, year 2009.

In one planet there are 2009 languages used. What is the minimal number of dictionaries necessary to translate from each language to any other? (Multilevel translation is allowed; each dictionary is used to translate only in one direction, e.g., from Latvian to Estonian, not vice versa.)

Example 2.

58th Regional Olympiad, year 2008.

Which natural numbers n can be written in the form $n = \frac{x}{y}$, where $x = a^3$, $y = b^4$, a and b – natural numbers.

59th Regional Olympiad, year 2009.

Which natural numbers n can be written in the form $n = \frac{x}{y}$, where $x = a^3$, $y = b^5$, a and b – natural numbers.

The main positive effects of repeated problems are:

a) teachers and students have main material for practice;
b) students are stimulated to study carefully the ideas considered in contests at least some years ago,
c) those who have been diligent are immediately rewarded,
d) the teachers are stimulated at least to open books/electronic files with the materials of last contests; having opened them, they often do not stop on a particular set of problems but continue reading,
e) the psychological pressure of the threat to remain with zero points in the contest doesn't exist more.

There are also some negative features of this praxis:

a) attempts of bringing the printed copies of last years' materials into the room where the contest is held,
b) reproducing the solution without proper understanding of it,
c) students fail while trying to identify the repeated problem and thus get confused and are not able to solve anything correctly.

These negative features have been minimized by strict rules while participating in olympiads. Students may use only writing tools. They are not allowed to use any paper, electronic devices etc. Students are not allowed to talk. Students may use WC only escorted by the person on duty.

To motivate students and teachers to prepare for olympiads A.Liepas Correspondence Mathematics School publishes many additional materials. Some courses in advances mathematics are held in Riga and in regions as well.

After the analysis of some results of the 2nd stage of the State Olympiad it becomes clear that the students who were preparing for this stage of the Olympiad and studying the repeated problem profoundly are awarded with good results and even diploma and price from the regional education department. About the half of the participants has solved the repeated problems good or excellent, besides the greatest part of these students has solved other problems also quite well.

We should also mention that there is no official declaration that there will be a repeated problem, but teachers have noticed that there is such praxis and inform colleagues and students.

So despite the negative aspects mentioned before it is good to have these repeated problems in the Olympiads because it is excellent opportunity for both teachers and students to prepare, to develop their mathematical talent and to be successful.

STUDENTS' ATTITUDES TO MATHEMATICS AT UNIVERSITY LEVEL

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Abstract

The aim of this article is to introduce a survey about the students' attitudes regarding mathematics at an Estonian university and present a synopsis of results relating to students' attitudes, motivations and their achievements in mathematics. The data was collected from first year baccalaureate students via a questionnaire which utilised a Likert-type scale and open questions. The educational implications of these results are also discussed.

Introduction

One of the strategic targets of Estonian higher education for 2006 to 2015 is to use higher education for the benefit of Estonian development and innovation. Scientific work and education is aimed at the needs of our society and economy. Estonian government has decided to implement a set of measures to promote the studies of science and technology. At the same time, the statistics of the state examinations reveal that the number of students who choose to take mathematics examinations is declining. The curriculum of mathematics has been simplified and confusion surrounding the whole issue does not seem to disappear.

In the 2003 TIMSS test our students showed excellent results in mathematics, gaining the 8th position among all the participating countries. In Europe, Estonia took 3rd place after Belgium and the Netherlands. 41% of our students judged their capability in mathematics "very good" (the international average was 40%) 32% considered themselves "average" (38%) and 28% "poor" (22%). Estonian students showed the lowest self-esteem related to mathematics among its group (Mere, Reiska & Smith, 2006).

Continually growing competition in a knowledge-based society has increased competition in universities. Universities are competing to acquire better students, more research scholarships, international students etc. The objectives of higher education have changed considerably. The prevailing approach has become client-orientated. On an institutional level, these aims are exploited through the "mission statement" of universities and on the level of study programs, through anticipated study results (knowledge, skills and values). The quality of teaching is closely related to the effectiveness and success of studies. Effectiveness depends on the aims of the subjects and the results. The most popular subjects at universities are those where mathematics are required as little as possible.

The success of mathematical studies at universities is influenced by emotional as well as social needs (Op 't Eynde & Hannula, 2006). Other important factors that have impact on the study of mathematics are attitudes, values and understandings (Hannula et al., 2007).

Motivation also plays an important role in the study of mathematics. The prominent question is why it is necessary to teach a specific mathematical topic and why it is relevant. The question becomes important when we are trying to find practical implementations. If concentrating on motivation, it is possible to find different ways how a specific subject influences us. In the classroom situation students are also affected by competence and social inclusion whereas research, understanding and socializing are opposed to rules, routine and mechanical lectures (Hannula, 2006).

Theoretical framework

The main focus of the research is on the study of mathematics at the baccalaureate level. Traditional theoretical lectures are often replaced by methods focusing on practicality and the topics are explained with the help of real-life situations. Mathematical applications have become more common during the course of studies in many other subjects such as economics, technology and science. Since the number of students with excellent skills in mathematics is declining, the effectiveness of teaching mathematics is becoming more important (Abiddin, 2007). Computers and mathematical programs at work have become commonplace and therefore the educational aims in mathematics have changed. Simple arithmetic has become less important and being able to form real-life mathematical models and using software has become more dominant (Petocz & Reid, 2006). The same topics are taught differently by different lecturers (Chval et al., 2008). Various possibilities to work out the criteria for high-quality lectures in mathematics have been proposed (Bergsten & Grenholm, 2006; Juter, 2005). Today's innovative lecturers have opted for new methodologies instead of using the out-of-date blackboard and chalk approach (Alsina, 2001). The purpose of the research is to examine the opinions of students and lecturers in order to increase the motivation among students to study mathematics.

Method

The view of mathematics indicator used in this research has been developed in 2003 as part of a research project in Finland. The statements in the questionnaire are grouped into seven topics (Rösken et al., 2007), which do not include motivation. I have used modified questionnaires to collect and analyze data from a sample of university students. In order to include motivation as a measure, we used Midgley's (2000) personal achievement motivation questionnaire. The mathematics-related beliefs questionnaire was developed at the University of Leuven, Belgium (Op 't Eynde & De Corte, 2003). The questionnaire on the characterization of mathematics was developed in Italy (Martino & Morselli, 2006). This survey was the pilot survey in

Estonia and the final survey will be carried out in autumn 2009 and 2010 in Estonia and Finland. We will cover samples which are drawn from first year baccalaureate students and their lecturers from two private and six public law universities. All of them will fill in a questionnaire in September 2009 when the students are starting their 1st year.

The statements in the questionnaire are grouped according to the following topics: (1) Experiences as mathematics learner, (2) Image of oneself as a mathematics learner and (3) View of mathematics and its teaching and learning. In the pilot survey, the students were asked to respond on a Likert scale (4 point, strongly disagree to strongly agree) for 66 questions. I also added open questions about their attitudes and motivations. The students were given 40 minutes to fill in the questionnaire and were told the questionnaires were anonymous. I used only quantitative method and collected 93 questionnaires from 2 groups: 1) students studying in Estonian language 2) full-time students studying in English.

Questionnaire

The objective of the pilot survey was to examine the reliability of the questionnaire. The data analysis is made in SPSS. I used maximum likelihood factor analysis as the extraction method. The eleventh-factor structure with loadings (loadings are presented in the brackets behind the statements) is presented below.

F1 Preparation of Lecture (Cronbach's alpha =0,433): How often does your lecturer give out additional lecture notes? (0,74); How often does your lecturer use real-life examples? (0,758) How often should there be practical tasks in your mathematics lecture? (0,686); There are enough study materials in mathematics (lecture notes, workbooks etc) (0,773); Mathematics lectures are too theoretical (0,747).

F2 Personal Achievement Goal Orientations - Performance-Approach Goal Orientation (Cronbach's alpha =0,870): It's important to me that other students in my class think I am good at my class work (0,900); One of my goals is to show others that I'm good at my class work (0,927); One of my goals is to show others that class work is easy for me (0,880); It's important to me that I look intelligent compared to others in my class (0,866).

F3 Personal Achievement Goal Orientations - Mastery Goal Orientation (Cronbach's alpha =0,749): It's important to me that I improve my skills this year in mathematics (0,811); I am very motivated to study mathematics (0,928); It's important to me that I thoroughly understand my class work (0,781); It's important to me that I learn a lot of new mathematical concepts this year (0,815); One of my goals is to master a lot of new skills this year (0,679); One of my goals in class is to learn as much as I can (0,886).

F4 Confidence And Students Beliefs (Cronbach's alpha =0,591): It is important for me to get good grades in mathematics (0,785); Usually students cannot understand mathematics, but only memorize the rules they learn (-0,798); Learning mathematics is mainly about having a good memory (-0,756); Getting the right answer in mathematics is more important than understanding how the answer was obtained (-0,825); If I cannot solve a mathematics problem quickly, I quit trying (-0,835); I am sure that I can learn mathematics (0,853); I think I could handle more difficult mathematics (0,872).

F5 Attitudes To Mathematics (Cronbach's alpha =0,734): Mathematics is a collection of facts and processes to be remembered (-0,754); Mathematics is about coming up with new ideas (0,876); I learn mathematics by memorizing and through repetition (-0,771); I usually understand a mathematical idea quickly (0,868); Mathematics is about solving problems (0,732); I cannot connect mathematical ideas that I have learned (-0,750).

F6 Enjoyment And Relevance Of Mathematics (Cronbach's alpha =0,661): Some knowledge of mathematics helps me to understand other subjects (0,849); Knowing mathematics will help me earn a living (0,782); I think mathematics is an important subject (0,817); Mathematics has been my favourite subject. (0,815); Studying mathematics is a waste of time (-0,859); Mathematics is a mechanical and boring subject (-0,919); I can use what I learn in mathematics in other subjects (0,798); Routine subject exercises are very important in the learning of mathematics (0,818); To study mathematics has been something of a chore (-0,846); I study mathematics because I know how useful it is (0,836); Mathematics enables us to understand better the world we live in (0,912); I can apply my knowledge of mathematics in everyday life (0,827); After graduating university I have many opportunities to apply my mathematical knowledge (0,872).

F7 Applications (Cronbach's alpha =0,735): Mathematical developments usually help to improve the economy (0,806); Knowledge of mathematics is important; it helps us to understand the world (0,887); Mathematics is useful for our society (0,885), After graduating university I have many opportunities to apply my mathematical knowledge (0,872).

F8 Student Competence (Cronbach's alpha =0,838): Mathematics was my worst subject in high school (-0,806); Mathematics is hard for me (-0,910); I am good at mathematics (0,912); I think it is interesting what I learn in mathematics (0,838); Compared with others in the class, I think I am good at mathematics (0,857); I understand everything we have done in mathematics this year (0,874).

F9 Teacher Quality and Role: (Cronbach's alpha =0,700): My lecturer explains why mathematics is important (0,815); We do a lot of group work during mathematics lectures (0,812); My lecturer doesn't let me solve only easy mathematical problems, but also encourages me to think (0,768); The lecturer has not been able to explain the processes we were studying (-0,863); My lecturer has not inspired me to study mathematics (-0,858); My lecturer tries to make mathematics lessons interesting (0,815); In addition to mathematics, the lecturer teaches us how to study (0,820).

F10 Cheating and disruptive behaviour (Cronbach's alpha =0,802)*:* I sometimes copy answers from other students during tests (0,927); I sometimes cheat whilst doing my class work (0,914); I sometimes copy answers from other students when I do my homework (0,862). I sometimes don't follow my lecturer's directions during class. Sometimes I disturb our mathematics class (0,805).

F11 Effort (Cronbach's alpha =0,637) I have to work very hard to understand mathematics (-0,908); I always prepare myself thoroughly for tests (0,911); I always carry out my home work diligently (0,862); I always prepare myself thoroughly for exams (0,818).

Conclusions

Based on the principal component analysis the structure of the first year baccalaureate students' views of mathematics is coherent with the structure from other researches. This gives a positive signal about the usefulness of the instrument, as the component structure remains stable in different populations. The pilot survey has been conducted and the results are reassuring to undertake the survey in autumn 2009 and autumn 2010 in two countries: Estonia and in Finland. This survey will be the first one which investigates the students´ views on mathematics in Estonia and Finland at university level. In further analysis, the relationships between students´ attitudes, motivations, beliefs and mathematical performance will be investigated and a comparison between the Estonian and Finnish students´ answers will be presented.

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BETWEEN EASY AND NICE, OR AGAIN ABOUT THE SAME: WHAT MIGHT BE REALLY ADDED TO THE ATTRACTIVENESS AND ACCESSIBILITY OF PROBLEMS

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I imagine that almost every person, independently from how educated she/he is or even how experienced she/he might be, has spent a remarkable part of her/his life moving successfully between these two milestones of human life mentioned in the title. Between easy (when everything is clear and every detail behaves predictably) and nice (when the circumstances are not so clear but situation as such seems to be challenging and inviting to our will and imagination). And we would like to make this principle of every activity (the one of problem solving no less than of life) clear.

In life it might be much more dramatic as it is in the problem solving, i.e. in the realisation of tasks. But already in that everyday process of exploring something, when only several numbers and only not so difficult ideas are the main players on the stage of mind and fantasy, there are some tricky scenes to be seen, to be understood and, of course, also to be enjoyed.

Let us describe some arithmetical adventure which might have been played anywhere, and let us raise some democratic questions which might be so naturally formulated in the form “Are all of them (the arithmetical adventures) of the same nature?”

There is a commonly known and brightly used answer whose meaning is more or less that even (any) silly person might produce much more (outstanding?) questions than any strong head would be able to answer. It is related to the fact, known by all who were in touch with exact thinking: each solved puzzle raises immediately at least nine new tasks, or widens the horizon. And that process seems to possess no end. Otherwise on one happy day all serious tasks would be exhausted.

Now we have the right and duty and are already expected to start speaking about the concrete things in order to be not so boring or, at least, we clearly indicate that we are not pretending to make an impression that we are producing some kind of big theory.

On the contrary, things, which we are now trying to touch, have the universal and very certain nature, and only from that point of view they might be interesting to modern people who have so many subjects to learn and so many ways to follow. Just as it is expressed in that poetical epitaph:

To follow you I'm not content.

How do I know which way you went?

Let us take the serious serial numbers, which were presented on the banknotes the author grabbed out from his pockets when producing these lines. The numbers found on those banknotes were 36959198, 43242954, 07311190, 91585787, 37809915, 01583416, 64671730, and 61817456. Seeing these integers we may state some circumstances which we could state even without undertaking any special action: e.g. that their digits are following one after another in chaotic unpredictable way; it would be difficult to expect that in any 8-digital number taken by chance all digits are different – and so it is in our case; some of integers contain zero digit, some don't, etc. As in life, so in arithmetic, we are sometimes eager to invent some interesting or otherwise curious classifying divisions of all objects into some (at least two) interesting parts. After the definition of such distribution we usually immediately start classifying all objects, we are in touch with, trying to understand in which of prescribed parts our elements are and how many of them there are in each of the parts. In arithmetic the set we divide so often is that of all positive integers. One might state that this is not always actual, or even that this is not so interesting, or even that this is not so challenging for the young generation. One might even believe that in the everyday life there is no place for infinite sets. One may think so. But the people in our everyday life sometimes (and not so seldom) have to deal with even much more complicated situations – not to speak about the distribution – just as it is reported in these paradox lines. Let us cite the immortal words of Mr. Harry Graham (1874-1936) giving that ideal of division into two parts the final poetical form and clear expression:

‘There’s been an accident!’ they said,
‘Your servant’s cut in half; he’s dead!’
‘Indeed!’ said Mr Jones, ‘and please
Send me the half that’s got my keys’.

Some remarkable and curious division of set of positive integers into two parts

We are going to introduce such a division in that set of all integers: take the number which is (usually) presented in its decimal form and try to get 0 from its digits using the parentheses and the signs of arithmetic operations. An immediate example in which we might be successful in achieving this is, for instance, a “regularly looking” number 1234 because $1 - 2 - 3 + 4 = 0$.

You may at once give the name to that clearly non-empty family – we will call such integers *zero’s friends.* So 1234 is *a friend of zero*. You will ask whether all integers are such *friends*. The answer is not because any of initial non-zero integers 1, 2, 3, 4, 5, 6, 7, 8 and 9 despite their neighbourhood with 0 isn’t *a friend of zero.* The first two digital integer 10 as well as its successor 11 clearly are *friends of* 0, then 12, 13, 14, 15, 16, 17, 18, 19 are not again. You may see that the numerals of our banknotes all *are friends of 0*, e.g. 36959198 is a friend of 0 because

$$((3+6) - 9) \cdot 5 \cdot 9 \cdot 1 \cdot 9 \cdot 8 = 0.$$

One might immediately make a lot of good common sense remarks concerning these numbers. One could start with an observation that huge numbers, or those which are so far from zero, have much more chances to become *friends of* 0 because if they have many digits so they have lot of chances that these four arithmetic operations applied together with parentheses involved will produce 0 from the digits. Or you may mark that if the number possesses two equal consecutive digits then it is also *a friend of zero*.

So we have a strong suspicion that any number starting from some (probably rather big) number is *a friend of* 0. How to prove it?

We can give you a good advice how to do it. Take sufficiently large integer having many digits. Then do the following. Subtract from the first digit the second one. The result can positive or negative. If the result is 0, we are done. Don't ask us why! We know you already know it! If the difference is still positive then subtract from that difference the following digits until you'll get the negative result. Then stop subtracting and start adding successive digits until the result will be again positive. Then start subtracting digits again and so on. We strongly believe that you understood where you must stop that process and how to finish the whole thing. Now use the pigeonhole principle.

Another interesting detail already after establishing the theoretical truth that there exists some integer *N* such that all greater integers ad infinity are *friends of zero* might be related to the interplay human being/computer and might be formulated in the challenging "the last of the Mohicans" form: find the greatest, or the last number *N*, which *isn't a friend of zero*.

Some Serbian problems unsolved in Lithuania

Let us take the following typical question, which was recently proposed at some Lithuanian competition and which is originally a Serbian problem or at least found by the author in the Serbian sources as a problem proposed at Serbian selection competition. The original question was to indicate the least multiple of 2009 with the sum of digits being clearly also 2009. For computer it might be and surely is an easy problem but for human being it proved to be a good exercise for the minds. By the way, it remained unsolved in the Lithuanian competition. It was structured in a standard way, firstly, asking for some such number (that is not necessarily the least), and so we get the problem which is lying in between easy/difficult problems stripe and this is very often so in the area of an easy/nice puzzle. So in that case we apparently have to deal with a serious challenge for the human mind (but not for the computer) and we may ask for the easy part to start in order to be able further in a good form and mood to return to that nice/difficult part. It is like as the warming up show in the popular music. As some introducing step there might be a proposal to find any multiple of 2009 with sum of digits also 2009. It is indeed rather easy – it is enough to find some multiple of 2009 with the sum of digits 2009 – $(2+0+0+9) \cdot 182 = 7$ or $2009 - (2+0+0+9) \cdot 181 = 18$. Since $2009 \cdot 9 = 18081$ possess exactly that sum of digits 18 then the number

1808120092009…..2009 (2009 is repeated 181 times)

will do (that modest number, as we see, possesses $5 + 4 \cdot 181 = 729$ digits).

Originally Romanian problem with unresistable determinant flavour

There are some problems which are formulated so that it is quite easy to make them more accessible or more difficult. Nice example of that kind is the problem proposed in the Romanian Olympiad. It ought to be told and repeated that the Romanian problems are proving the astonishing degree of almost metaphysical beauty. Often they are also as difficult as they are nice – and nice they always are. The problem, which is cited below, is not as difficult as some Romanian problems prove to be – but they also are usually elegant. In that problem you can feel some irresistible flavour or reminisces of determinant theory – especially if you start solving it.

The elements of the set $M = \{1, 2, 3, \ldots, 98, 99, 100\}$ are arranged in a 10 x 10 table as below:

$$\begin{array}{l} 1\ \ 2\ \ \ldots.\ 10 \\ 11\ 12\ \ \ldots\ 20 \\ \ldots\ldots\ldots\ldots\ldots \\ 91\ \ 92\ \ \ldots.100 \end{array}$$

Prove that no matter how 10 numbers are removed from the table, the remaining 90 contain a 10-element arithmetical set – that is the arithmetic progression.

It might be mentioned that it would be possible to formulate that problem in the psychologically more challenging way asking in the sense of ”Prove or disprove the fact that no matter how...”. Another interesting circumstance is that square ordering of these 100 numbers. Then if we want to reduce the problem but are still leaving the whole intrigue it would be so natural to reformulate that problem with the first 9 integers ordered naturally in a 3 x 3 square. After that you may take the square of any size – it would be all the same! But not psychologically.

Reformulating the problems

If you want to propose the problem for Mrs. or Mr. Smith and any of them suddenly sees their own name mentioned in the text of the proposed task then psychologically that proposed task might appear to them not so difficult as it originally had been. Because the notion Smith is for them more understandable the rest might appear for them already more attractive or, at least, quite acceptable.

Let us take some problem from the Russian Olympiad, which is already attractive in its content and elegant in its form. That problem belongs to the famous Russian problem author I.S. Rubanov. Below you see it in its original formulation.

7 cards with numbers 0, 1, 2, 3, 4, 5, 6 are given. Peter and Basil make moves in turn taking one card by each move; Peter makes the first move. The player who can construct of his cards a decimal number divisible by 17 earlier than his opponent is declared as a Winner. Determine which of two players has a winning strategy.

The nature of the problem clearly implies that if any of the players is able to possess the winning strategy then that is exactly the first player. So to say, if any of them is able, then any means the first. Now to the details. It appears that there are 3 of them – all easy to describe. The first detail is: if being first I take the card 1, then you must necessarily take 5. Otherwise I'll take it at my second move and will present the number 51. So if I take 1, so you must take 5. Then my second turn is to take 3. Then you are forced to take 4, otherwise I'll take 4 and will show you 34. So I've already chosen the numbers 1 and 3 while you have 4 and 5. Now the last stroke, which will clearly break camel's back, will be the selection of 6 and presentation of number 126, which is clearly $8 \cdot 17$.

You may compare the text of the problem with the author's presentation of the same problem and not forgetting that the last IMO happened in Bremen (Germany). It was proposed in some Lithuanian competition for younger grades 7 and even 8.

On silent winter nights when last lonely passengers disappear from the streets the eternal Patron of Bremen city Roland is climbing down from his monument and together with Maestro Cat starts arranging, as they call it, *the silent Bremen-17 gam*e. For that, 7 cards with numbers 0, 1, 2, 3, 4, 5, 6 are necessary with the only number on each of the cards. Roland and Maestro Cat are taking in turn the cards one by one; Roland usually starts first. The player, who is able using his cards to present a number divisible by 17 earlier than his opponent, is declared to be a winner. In the news portal www.ihaha.com there were furious quarrels concerning the fact whether any of them is indeed able to turn these cards in such a way that he would always be able to win independently of what his partner undertakes.

There appeared one wiseacre whose name was Maestro Rex and who was constantly running about claiming that:

(A) if any of them can really win independently what his partner is ever going to undertake then that person is that who starts first.

Is maestro Rex right? Explain your motivation.

(B) So how is that: can indeed some of these persons win independently of what his partner is going to undertake?

The author spent many years solving and teaching how to solve problems. Many other insights in that direction, due naturally also to many other authors, with which the author was always eager to discuss and cooperate, might be found in his common publication (Barbeau & Taylor, 2009) which recently appeared as a Chapter I in the Springer text.

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ALTERNATIVE INTERPRETATIONS AND USE OF THE TIMES COMPARATIVE IN EVERYDAY LANGUAGE AND SCHOOL MATHEMATICS: PRE-SERVICE TEACHERS' VIEWS

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Abstract

In many everyday and even in scientific contexts the following kind of expression is used: 'the value of quantity A is N times larger (or smaller) than the value of quantity B'. We will here refer to this expression by the name 'times comparative'. Because future mathematics teachers are in a key position for giving attention to this kind of expressions in school mathematics, we were interested to know, how well informed the pre-service teachers were of the use of the different alternative interpretations of the times comparative and should this issue be brought up or not in school mathematics. The article presents the main results of the pilot study and discusses the significance of examining the times comparative in school mathematics.

Introduction

Multiplicative comparison, which is on the focus of this article, has been traditionally introduced in Finnish mathematics textbooks after basic multiplication and division, and in preparation for the conceptual analysis of *ratio* in the last two grades of primary education. This is understandable, because the assimilation of the concept of ratio requires multiplicative comparison in which the focus is on the multiplicity of two numbers or two quantities ((see e.g. Greer 1994, Tourniaire & Pulos 1985, Vergnaud 1983). When lengths are examined, for instance, the *result of the comparison* can be expressed *accurately* in the form '*the length of Y is N times the length of X*'. We can also say '*Y is N times as long as X*'. Synonymously to these expressions, one often sees both in general language and in other contexts the expression '*X is N times longer than Y*'. Below, a structure of this kind is called the *times comparative*.

In the current study, the *times comparative* is thought to have at least two possible main meanings, of which the *purely multiplicative meaning* was already discussed above. In this interpretation the expression '*N times bigger'* means the same as '*N-fold'*. The expression '*N times smaller than*' means the same as *'the N-th part of something'*. On the other hand, the expression 'Y is N times bigger than X' has been thought in some contexts to mean the same as the expression 'add N times the quantity to the value of X'. This interpretation combines in a way the additive comparison and multiplicative comparison. Therefore, this interpretation of the times comparative is called below the '*combined meaning*'. Using this meaning, the result of the reasoning or calculation is thus (N+1)*X rather than N*X. In N is small the results can be substantially different. The context may indicate which meaning was used or is referred to, but the ambiguity of the times comparative in this sense still remains.

As for school mathematics, times comparatives are found in modern Finnish textbooks only sporadically. Thus teachers might not discuss the interpretation of the expression at all in the classroom. The situation was different in the 19^{th} century textbooks and even in the 20^{th} century all the way to the 60's: the times comparative was commonly used in textbooks in parallel to the other types of expression, and its use was in accordance with how the expression is used in everyday language. The standpoint taken by the authors changed by the 1950's and 1960's. They started to view the times comparative as a problematic issue, and its meaning was typically taught as follows (translated from Finnish): "60 m is not three times more than 20 m. It is only two times more" (Kuuskoski & Stara 1961, 117). In other words the purely multiplicative meaning was explained to be erroneous. The same was also the case in Hallonblad's (1961, 101) textbook in which it was urged that the expression should be used in the 'proper' combined meaning. An effort was also made to eradicate the times comparative altogether (again translated): “You shall not say that 8 is ‘twice more’ than 4” (Vahervuo 1954, 74, see more examples, Kohonen 2006). It is not clear if the textbook writers of those times really knew what kind of linguistic change they were advocating, and how difficult in general it is to change established structures in language. On the other hand teaching against general usage of language – whenever it has been practised – has, however, given rise to repeated controversy and uncertainty.

Research questions of the pilot study

Because the future primary and secondary school teachers are in a key position for taking into discussion times comparatives in school mathematics, we were at first interested to know which interpretations they themselves would use in solving word problems involving times comparatives, and how consistent they are in this respect. Secondly we wanted to know how well informed they were of the use of the different alternative interpretations of the times comparative in different contexts. Thirdly we asked their opinions on whether and why these expressions should be given more attention in school mathematics. At which class and within which topic could this be done?

Method

The research data on prospective mathematics teachers and primary teachers was acquired in connection with the mathematics didactics courses held in the Department of Educational Sciences and Teacher Education at the University of Oulu in the spring term of 2007. 30-40 minutes were spent by each group of teacher trainees to complete the questionnaire test designed for the purpose. Issues connected with the contents of the test were not discussed in advance in any of the groups. The mathematics teacher candidates (n=21) had already studied mathematics for several years, while most of the primary teacher candidates (n=56) were participating for the first time in a university course on mathematics and its teaching and learning. On the day when the tests were carried out, three primary teacher students were absent. Among the prospective mathematics teachers, 60 % of the entire group participated in the test.

The questionnaire

Originally the questionnaire consisted of four sets of problems. We will examine here three of them. In the first section, the subjects had to solve five word problems chosen or slightly modified from mathematics textbooks for primary education. Each problem involved a times comparative of the type ‘N times more (less, bigger, larger) than X’. The purpose was to find out which interpretation of the times comparative mostly formed the basis for the students’ solutions, and how consistent the use of each alternative meaning was in each group of students.

The second section included ten statements picked out from different types of texts, each one of which contained one times comparative. The students had to evaluate which alternative interpretation each of the statements referred to. For instance, the second one of the statements picked from a newspaper was as follows: “Houses are now built two times more than three years ago”. The alternative answers were these: (a) (approx.) twice the number, (b) (approx.) three times the number, (c) other. In addition to newspaper samples like this on everyday contexts, the section also included samples from both administrative language and scientific contexts.

In the third section the students were asked about their conception of the use of the times comparative in school mathematics. Should this issue be brought up or not? If there is a case for paying attention to this issue in teaching, then in which grade and in connection with which mathematical contents should it be done? The answers to these questions were classified depending on the data.

Results

According to the data most of the students based their solutions to all five word problems on the purely multiplicative interpretation of the times comparative. Three students were found who used the combined meaning in all the problems. So these students identified the relation 'Y is N times more than X' with the relation 'Y is N+1 times X'. Upon finding that even the rest of the students participating in the test used the same meaning basis in at least three of the five problems, adherence to the use of the chosen interpretation throughout the set of word problems was highly consistent in most of the cases.

As for the second section, most of the subjects (75 %) thought just rightly that the writers of all the ten text samples had used the times comparative in a purely multiplicative sense. A clear minority (25 %) of the subjects considered that the writers had used the combined meaning or some other meaning in the text samples.

As for the third problem the teacher candidates thought nearly unanimously that the various alternative interpretations and uses of the times comparative should be considered in school mathematics. General arguments in its favour included the opportunity to specify or clarify the use of mathematical language and to direct towards proper usage of language in multiplicative comparison, thereby avoiding conceptual confusion. Only five students (6 %) thought that the times comparative should not be discussed in school mathematics. Their argument was that an analysis of the use and alternative interpretations of the times comparative would only "confuse the pupils' heads". Three of the students (4 %) did not report their viewpoints on this issue.

Those who had a positive attitude towards the analysis of the times comparative in school mathematics had quite varied ideas of when and in which connection the issue should be brought up. Most of the primary teacher candidates (93 %) who indicated their viewpoints considered the lower primary school grades from 1 to 6 to be the most appropriate stage. Some of them (23 %) even thought that the issue could be introduced in grades 1 to 3. Among the mathematics teacher candidates who responded, the majority (59 %) considered grades 7 to 9 to be an appropriate time for considering this issue.

As for the mathematical context in which the times comparative could be appropriately discussed, fifteen (27 %) primary teacher candidates failed to provide a response. Most primary teacher candidates mentioned multiplications and divisions (39 %) and comparisons of the magnitude of quantities, measurement and geometry (34 %) as appropriate mathematical content areas. Only three of the primary teacher candidates mentioned ratio, proportionality and percentage calculations. Five of the prospective mathematics teachers (29 %) who responded mentioned that ratio and proportionality was an appropriate mathematical content area for discussing the times comparative. References to multiplication, division and measurements were made in twelve responses (62 %). Four students to become mathematics teachers did not provide an answer to this question (19 %).

Conclusions

The future primary and secondary school teachers and their education are in a key position for taking the times comparative into discussion in school mathematics. As we have seen, the problem is not motivation but rather weak advance information about the alternative interpretations of the times comparative and about the fertility of this topic in school mathematics. On the other hand, future primary and mathematics teachers do not seem to have a very clear and uniform view of the prevalence and the use of times comparatives in different contexts. While most of the subjects used consistently the multiplicative interpretation of the times comparative in the solution of word problems in mathematics, some of them thought that in everyday language and in the language of science and mathematics the meanings are different. In particular, they often thought that when moving from the spoken language towards more formal language of administration and science, the relative prevalence of the pure multiplicative interpretation would diminish in favour of the combined meaning while this was in fact not the case in reality. Therefore we cannot really think that without adequate information they could give their pupils a valid picture of the issue in a classroom situation, either. It also became evident that only some of the subjects had a valid view of the grade and mathematical content in which these expressions could and should be discussed.

However, these deficiencies could be easily remedied by introducing the topic in the mathematics studies of didactics in teacher education. In fact according to our previous teaching experiences from the courses of mathematics didactics, the topic will give rise to a vivid discussion. Further the topic brings school mathematics closer to everyday mathematics, and it may deepen proportional reasoning and knowledge about ratio. How this kind of discussions really might affect pupils' arithmetic skills and knowledge, could be one of the research topics in future. Further it would be important and interesting to know, which kind of effects times comparatives do have to the problem solving processes in different kind of multiplicative and proportional problems.

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Textbooks and dictionaries

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THE NUMBER LINE AS A TEACHING AID IN THE GRADES 1-2: TEXTBOOK ANALYSIS AND PUPIL INTERVIEW

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Abstract

The number line is mentioned as one manipulative among others in the Finnish National Core Curriculum (2004) on first two grades. Some authors of mathematics textbooks have interpreted this in such a way that the number line is actively taken in use from the first days of elementary mathematics. Because there are some contradictory views of the use and role of the number line among mathematics educators and teachers, the authors wanted to find out what the situation is by using pupil interviews and making a textbook analysis. The interviews were made for second graders at Oulu University School February 2009. The test was based on rational analysis of text book exercises. It shows that although the second graders understand the basic ideas of the decimal system and basic calculations, they can have great difficulties in both measurement and number line tasks.

Introduction

The historical starting points for development of different number systems, like the decimal system, lie apparently in the counting and comparison of discrete sets (Barrow, J.D. 1999, 2.chapter, 49-151, Wilder 1973). This seems also to be true from the psychological perspective. The six-year-olds can list the numbers from one to twenty with ease, they can count the number of items in a discrete set, and they can compare discrete sets (Aunio 2006, Gallistel & Gelman 1992, Hartnett & Gelman 1998). On the other hand there are clear deficiencies in length measurement and comparisons of lengths. The same implies the use of the number line where children spontaneously notice the points themselves instead of their distances of from the zero (Fuson 1984, Ernest 1985, Petitto 1990, Ryan & Williams 2007, 87-99).

From the pedagogical point of view the cardinal aspect of the number is emphasised in the school starters' mathematics. The logico-mathematically more difficult idea of continuous quantities, like length or mass, is traditionally taught after the discrete number concept and number symbols in the Finnish textbooks, usually on first grade spring. Now into this widely approved order of teaching has been introduced a new additional factor, the number line.

In fact the number line is mentioned as one manipulative among others in the Finnish National Core Curriculum (2004) on first two grades. Some of the authors of the textbooks for grades 1 and 2 have interpreted this in a way that the number line has to be present from the beginning of the first grade. In these books it is used connected to the learning of the number concept and basic numbers 1-9 and 0 (cf. Oikkonen-Sotka et al. 2006). Some of the authors have gone even further. Children are expected to show basic calculations like 5+3=8 or 8-5=3 with arrow diagrams on the number line and vice versa (cf. Haapaniemi et al. 2006).

In which way the number line has been introduced in the Finnish mathematics textbooks? What kind of knowledge the pupils have to possess for understanding the basic ideas of number line and use of number line diagrams in modelling basic additions and subtractions? Do even second graders have enough knowledge to solve some basic number line, measurement, and modelling problems picked from the Finnish textbooks of the first and second graders? These are the main problems that will be discussed in the following using a textbook analysis and an empirical study.

About the use of the number line on primary level mathematics
The number line on first grade textbooks, a rational analysis

From mathematical point of view the number line is a geometric representation of the real numbers. The place of each point can be uniquely expressed by a given number and conversely. In teaching and learning the basic ideas of number line it would be essential to pay attention to the lengths of the intervals and to meanings of the numerals marked on the number line. Furthermore because a concrete and finite representation of a number line is a ruler, it would be very natural to start from the use of it the introduction of the number line.

In which way has the number line in fact been introduced in the Finnish mathematics textbooks? Common feature in the books is that the measurement of length is left to the first year spring clearly after the presentation of the number line. What's more, the basic ideas of the number line are left for the teacher's guide. For example on the third double page of Tuhattaituri 1a book (Haapaniemi et al. 2006, 10-11) the pupil is advised to connect a picture with numerical information to the number line. On the inverse problem type the pupil has to draw “correct amount of circles” to correspond the points on the number line. The obvious purpose of this double page is to familiarise the pupils with the number line and connect the cardinal and measurement meanings of the natural numbers from the beginning of the first grade. Unfortunately these exercises can be solved simply by finding the correct number symbol from the number line and connecting the two presentations that way. Hence it is possible that no connection between the cardinal and measurement conceptions of the number is made. It is justified to argue that this kind of an exercise enhances the sequencing skills which have been learned with the discrete sets. A better presentation for this scheme might be a number track (Bottle 2005, 97-98). It is also worth mentioning that if the number track is started from zero like in the Tuhattaituri 1a book (ibid. 2006, 18), it might confuse the correspondence between the number symbol and the number. It is worth mentioning that other first grade books do not take the number line into action at all (cf. Putkonen et al. 2005, Rikala et al. 2007) or they just show a continuing number line when teaching the basic numbers 0-9 (cf. Okkonen-Sotka et al. 2006). Another thing worth mentioning is that a number track starting from zero is not introduced in the other books. This seems to be a reasonable solution for avoiding confusion.

Using the number line on basic calculations: task analysis

As stated above the number line has been taken into use in the most of the Finnish mathematics textbooks already at the first grade. One textbook goes even to the use of the number line for teaching addition and subtraction (Haapaniemi et al. 2006). To solve the addition problems the child is advised to "Start the arrow from the first addend. Then move to the right as much as the second addend shows and you will get the sum."

This advice is very general when thinking of all the knowledge and skills needed to solve the problem with understanding. In fact to translate a numerical expression like 3+2=5 into a number line representation needs at least the following knowledge:

(a) the knowledge that a number is represented by an arrow of the given length,
(b) the knowledge that the first arrow to be drawn begins at the origin (compare to the length measurement)
(c) the knowledge that the beginning of the second arrow is where the first arrow ends (compare to the length measurement)
(d) the knowledge that the combined length of the two arrows represents the sum of the addition. (cf. Ernest 1985, 417-418.)

In addition to these also a skill to draw the arrows of given length is needed. To succeed in this is not so clear either for the first graders.

What is stated above is also true for subtraction, for example 5–3=2, in number line diagram. In the same textbook the children are asked to:

“Start the arrow from the minuend and move to the left as much as the subtrahend shows and you will get the difference.” The sentence “move to the left as much as the subtrahend shows” is not clarified in this example either. Also what is missing is the arrow representing the minuend which should be reversed to the one representing the subtrahend. The divergent direction of the arrows is the main problem when representing the subtraction on the number line (Hart 1981, 87). This problem is especially evident in the exercises where a number line presentation has to be transformed into a number presentation. If and when the arrow representing the minuend is not present as in the textbook Tuhattaituri 1a (ibid., 71), it can be very hard for the pupil to see what is subtracted from what.

So as argued before when basic additions and subtractions are represented geometrically on the number line, there is need for both common length measurement skills as well as specific knowledge about the connections between the symbolic and number line representations. If pupils fail on the number line problems it does necessarily mean that they fail in understanding the addition or subtraction. The problem can be that they do not yet have the specific knowledge and skills needed for presenting numeric information on the number line and vice versa (Ernest 1985, 418-420). If there is no teaching of length measurement and detailed analysis of the use of the arrows, the possibilities that the pupils can understandingly use the number line are low and the solutions in a way or another incorrect.

Empirical part of the study

Research problems and method

From the prior researches and from the point of view of the rational task analysis above there is every likelihood that even the second graders do have problems with length measurement, in understanding the idea of the number line and presenting addition and subtraction with arrows on the number line (Carpenter et al 1981, Ernest 1985, 419, Hart 1981, Ryan & Williams 2007, 91-95, 183-190). One of the main problems of the empirical part of the study was to find out, what kind of arithmetical and measurement knowledge do second graders have to spontaneously manage arithmetic on the number line. It is very self-evident that if the second graders have problems in this, the first graders doubtfully manage any better. Second essential research problem was to find out how the second graders can shift from the arithmetic sentences to the number line diagrams and vice versa. Third problem was to find out what kind of strategies, incorrect and valid, do the interviewed pupils use in the length measurement and number line problems.

The data was gathered in Mathematics Didactics course in the second grade of Oulu University School (n=21) in February 2009. First year student teachers were the interviewers (n=21). All the interviews were conducted using principles which were agreed beforehand. One of the authors planned the used test with the student teachers and oversaw the interviews in situ. Majority of the pupils used much less than 45 minutes for the test. The test consisted of 8 tasks and 2 extra tasks. The class was using Laskutaito 2b book (Rikala et al. 2003) in which the use of the number line is minimal. For example basic additions and subtractions are not done with the number line at all in the book. The measurement pages of the book were still to come for the pupils. On the other hand a lot of arithmetic including the structure of the decimal system, the comparisons of numbers and the arithmetic operations with numbers 0-1000 were familiar to them.

The test used in the interviews

The test consisted of four parts. The first part handled the sequencing of numbers 0-1000 (task 1, two sequences) and the structure of the base ten system (task 3, two parts). The second part consisted of length measurement. In the first task the pupils were to measure the length of an object using a broken ruler (ruler starting from 5cm and ending to 22cm, the length of the object 11cm). In the second task the pupils were to measure the lengths of four different length strips using a line of blocks drawn on top of them (task 6). This exercise was formed using a task from Laskutaito 2b book's measurement part (Rikala et al. 2003, 68). The third part was about the pupils' number line knowledge and skills. In the first task the pupils were to find and mark four numbers on the number line (4, 8, 12 and 17). Points were marked and the numbers 0, 5, 10, 15 and 20 were attached to the points. In the second task (three parts) the pupils were to mark the place on the number line where "Kassu the smiley face jumps the length shown in a die." The idea and form for this task is from Tuhattaituri 1a book. The task was made harder by starting the jumping from another point than the origin (cf. Haapaniemi et al. 2006, 12-13). The third task (four parts) tested the pupils' skills to show the basic additions and subtractions on the number line and vice versa. The idea for this task comes also from Tuhattaituri 1a book. The arches were substituted by arrows and the calculations were in range of 0-20 (cf. Haapaniemi et al, 2006, 54-55, 66-67). The fourth part of the test included two extra problems not directly related to the number line. So we do not consider them here more exactly.

Main results

The results are very similar with the results of the national survey in USA in the beginning of the 80's (Carpenter et al. 1981, cf. Ernest 1985, Table I, 419). Where the number sequencing skills, the decimal system and the arithmetic operations with numbers 0-100 were fairly well known, there were clear mistakes and thinking errors with the more demanding measurement and number line problems and in representing the operations on the number line. The results are summarised in the following Table 1.

Clearly the most difficult were the problems about length measurement and those measuring the understanding of the number line and using it with the basic arithmetic. Only five pupils (24%) could do measuring with the broken ruler correctly. Almost half of the pupils noticed the points and counted simply their number starting from 5cm and hence ending with one unit too much i.e. 12cm (44%). The rest had only noted the endpoint and gave 16cm as the answer (32%). The measurement with the blocks was considerably easier than the one with the broken ruler (67% correct). Still some of the pupils had noticed just the points or the starting point of the measurement was neglected (20%-24%).

Finding the point from the number line was an easy task. The pupils just searched for the number symbols and circled them using the nearest number as a starting point. This showed also that there was no need to understand the measurement idea of the number line in order to solve this kind of problems. This understanding was however needed in the jumping squirrel task as can be seen from the success rate. Because the starting point for the jumping in all the three problems was greater than zero, it could be anticipated that at least some of the pupils start from the beginning of the number line.

This happened with more than one third of the students in all three parts (38%). In the third part there was an additional problem because the starting point was in the middle of points 2 and 3. Only two of the pupils could note this and got the right answer. Majority of the pupils marked six instead of the correct six and a half (52%). Interestingly three pupils marked four and a half. Apparently they had understood to look at the intervals but still had started from the origin of the number line in all of the three parts.

As assumed representing the calculations using arrows on the number line and converse was too hard for majority of the pupils. The calculation 9+7=16 was correct in 38% of the papers. Majority of the pupils marked just the point marking the first addend or the sum. There were clearly problems also in the reverse tasks. Only 38% managed to do the problem from the number line to symbolic form 4+8=12. Most common mistake was to write 4+12 or 4+12=16 (24%). Judging from the answers the pupils noticed the end points of the arrow instead of the length of it. The subtraction was even harder (33% correct). The most common errors were 14+22 or 14+8 (29%). Again judging from the answers the pupils noted only the endpoints not the length of the arrow not to even mention the direction of the arrow.

Part/ Task	Success %	Failure %	Empty %
Part I			
Number sequence (2 ex.)	95	5	0
Decimal system (2 ex.)	80	20	0
Calculations 3+4=? and 9+7=?	95	0	5
Part II			
Measuring with a broken ruler	24	76	0
Measuring with blocks (4 ex.)	67	33	0
Part III			
Point on a number line (4 ex.)	100	0	0
Moving on a number line			
all 3 correct	10	90	0
at least 2 correct	63	37	0
at least 1 correct	71	29	0
9+7=16 on a number line	38	62	0
From the number line to 4+8=12	38	62	0
From the number line to 22-8=14	33	57	0

Table 1: Success percentages and percentages of the thinking errors in different tasks (n=21)

Concluding discussion

Both the results from the pupil interviews and earlier studies show consistently that using the number line in grades 1 and 2 may increase challenges and result learning difficulties. It seems obvious that even the second graders spontaneously focus on the points instead of the lengths of the intervals on the number line. This could be seen in the tasks where the pupils were asked to move an object on the number line by a certain length. The second graders focused attention on the points instead of the amount of movements in intervals. Similar was the situation with measurement tasks where there was no possibility to align the starting point of the object and the ruler. Majority of the second graders noted only the endpoint of the object and counted the number of the points. This shows that in order to learn with understanding the basic ideas of the number line, the counting schemes learned with the discrete sets should be directed to calculate the intervals instead of the points. This does not seem to happen spontaneously. So using the number line without teaching firstly the basics of length measurement will lead easily to the rote learning and point wise thinking instead of thinking by the lengths of the intervals (Carpenter 1981, Ernest 1985, Fuson 1984, Petitto 1990, Ryan & Williams 2007, 91-99). As for the learning of basic additions and subtractions, it is very hard to understand, why to make this process more difficult by relying heavily on number line presentations even from the beginning of the first grade. The connection between the cardinal and measurement meaning of the number is needed, but it does not mean that the number line should be taken in action when teaching basic calculations (cf. Ernest 1985, 422-423).

As a conclusion it can be said that the use of the number line from the beginning of the first grade seems to be very early and vaguely justified. If the children have to master number line presentations, teachers must ensure that they are explicitly taught the basic ideas of the number line and the use of the model. Without that children will easily lead to use the discrete, point wise thinking instead of focusing on the lengths of the intervals. Moreover transferring the arithmetic sentences into the number line diagrams and reverse will easily lead to the rote learning. Because most of the first and second grade pupils seem to learn decimal system and basic arithmetic operations without the use of the number line, it seems to be a bad idea to interfere this process with number line diagrams in the first two classes. If and when the number line model is taken into use, the most suitable place might be in context of fractions and decimal fractions. In this context the number line model can show the interrelations of fractions and decimal fractions and form a base for expansion of the numbers up to the real numbers.

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POSSIBILITIES OF USING MATHEMATICAL MODELS IN TEACHING STUDENTS

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Abstract

Study of conflict situations is the aim of game theory. Mathematical game theory is a theory of mathematical models for decision-making in case of interest conflicts, uncertain situations and risks. The aim of using game theory with school students is facilitating their rational behaviour skills in conflict situations, and choosing optimal strategy.

In a whole series of problems it is necessary to analyse situations, when interests of two or more contending sides meet, and each side pursues its objectives and the result of any action thereof depends on the "opponents'" actions. These are called conflict situations and are studied by game theory. Game theory is a theory of mathematical models for decision-making within a conflict. In contrast to a real conflict a game has certain rules. The sides participating in a conflict are called players. A game, which is played by two players, is called a two-person game. If the number of players is greater than two, it will be a multiple game. A game, in which the benefit of one of the players is equal to the loss of the other, is called a zero-sum game. Combination of rules, which determine the choice of player's actions at each turn depending on a situation, is called a player's strategy. If there are two players, the combination of all benefits can be assumed as a gain matrix (payoff matrix), which determines, what payoff should be made by one player to the other. Usually it is considered that rows of the matrix correspond to the first player's strategies, and columns – to the second player's strategies. Depending on a possibility of preliminary communication there are cooperative and non-cooperative games among players. A game is cooperative if players settle their strategies among themselves before the game starts. A game is non-cooperative if players cannot synchronize their strategies.

Let's consider a typical conflict situation. Two friends, a boy and a girl are arranging their weekend. The boy suggests going to the hockey, because it is going to be an interesting game, but the girl wants to see a play in the theatre. They have been friends for a long time, and if they go to their favourite place alone, it will not give pleasure to any of them; therefore, it is better to go together. If they both go to the theatre, as the girl wants, she will receive the maximum of pleasure (2 units). The boy does not want to go to the theatre, but the girl's presence at the play will smooth over his pastime and he will receive 1 unit of pleasure. If they both go to the hockey, everything will be the opposite way: the boy will receive 2 units of pleasure and the girl – 1 unit. If each of them go to the favourite place alone, let's assume that they receive negative pleasure (-1 each). The same will happen in the impossible case, when the boy goes to the theatre and the girl to the hockey. Thus, it is important for them to spend the evening together. A cooperative game with a bimatrix looks like this (fig. 1).

		A girl	
		The hockey	The theatre
A boy	The hockey	(2,1)	(-1,-1)
	The theatre	(-1,-1)	(1,2)

Figure 1. A bimatrix of a conflict among friends.

A bimatrix element (a_{ij},b_{ij}) shows the benefit of the corresponding choice for the boy (a_{ij}) and the girl (b_{ij}). How should they act?

In cooperative games players usually act according to the coordinated common strategy. Pure common strategy is indication of players' common choice of any bimatrix element. Common mixed strategy is a distribution of probabilities on the set of bimatrix elements. The discussion between the boy and the girl actually comes to finding the optimal mixed strategy for them ($p_{11},p_{12},p_{21},p_{22}$), in which p_{ij} is a probability describing game participants' common choice of a payoff bimatrix element, which would satisfy both of them. If friends keep to different uncoordinated mixed strategies then in a system of coordinates for payoff values the set of possible payoffs makes a triangle ABC with its vertices at points (-1,-1), (1,2), (2,1) (fig. 2).

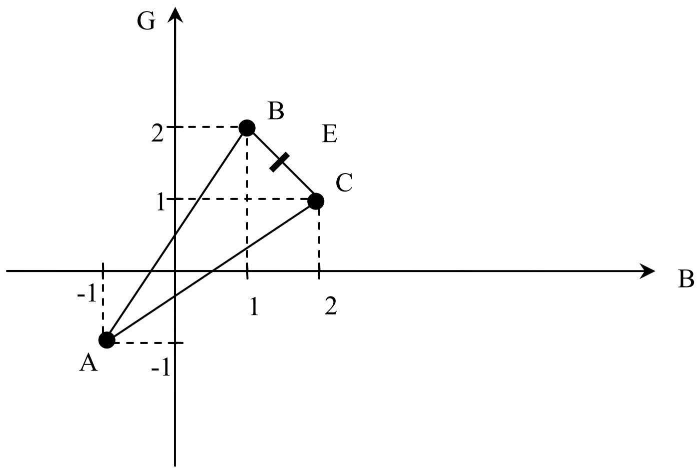

Figure 2. Solution of a cooperative game „Discussion among friends".

Any point of the side BC of the given triangle can be considered as solution of the studied discussion among friends. BC is the Pareto set – a set of optimal solutions of the conflict: along this line the increase of the girl's pleasure received from the event is possible only due to the decrease of the boy's pleasure (Замков, 1999). A possible solution of this conflict with equality of the participants: with a probability of ½ the boy and the girl go to the hockey and with the same probability they go to the theatre. In case of equality of participants the solution will be at the point E in the middle of the side BC: E (3/2, 3/2). The average payoff of each participant is 3/2. The most rational behaviour of the players in such situation is agreement to visit sporting events and theatre performances together in turn. Rational solution of the given dispute confirms the importance and necessity of finding compromises in various real-life situations.

Often players make their choice depending on the received information about actual circumstances in the development of the conflict. In such situations positional games can be used. These are coalition-free games, which model players' sequential decision-making processes in conditions of time variant and incomplete information. Here positions are states of the game, alternatives – available choice in every position. For visual presentation of a positional game a scheme "tree (graph) for solution" is usually used. In positional games with complete information a player, before his turn, knows his position on the game tree. In games with incomplete information a player, before his turn, does not exactly know his position on the game tree (Аронович, 1997). Positional games can be reduced to matrix or bimatrix games. Positional multistage two-person games are chess and draught.

As an example a positional game for making decision about team's participation in a sport tournament can be considered. There is a basketball tournament in a college among student teams from different years. A second-year students' team B, as coaches think, can be regarded as a strong competitor to the fourth-year students' team A, which is the favourite of the tournament. However, some of the players from the team B are on the sick list. Strategies of the team B: to take part in the tournament (B_1) or not to take part in the games (B_2). In its turn, the team A settles a question about inviting some experienced players from young college staff (accepted by the rules), in order to keep its leadership. Strategies of the team A: to invite experienced players for strengthening (A_1), not to invite players from outside (A_1). Graphically this can be represented as a "solution tree" (fig. 3). Payoffs of the teams in the figure 3 are shown in brackets: the first number is a relative payoff in "points" of the team A, the second number is that of the team B.

The team A, as the favourite of the tournament, can get maximum of 10 points if the strong opponent (the team B) does not take part in the games and with the help of the invited players. If the team B takes part in the games, the team A will get only 8 points even if it is strengthened by the invited players. If there is no strengthening, both teams can get equal number of points (6), without the strong opponent (B) the team A will get 7 points. The points that teams A and B will really earn at the tournament can also be different because it will depend on the play of other teams.

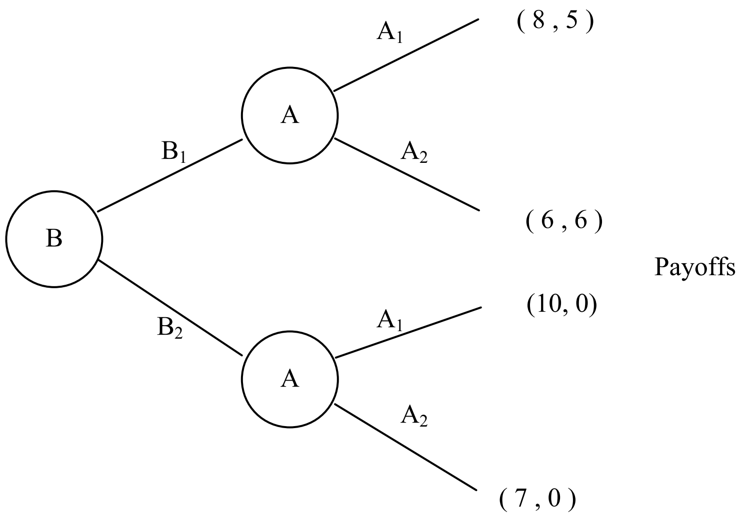

Figure 3. Tree for solution of a positional game.

Therefore, the given payoffs are most probable according to estimations of the coaches. This positional game can be reduced (normalized) to a bimatrix game (fig. 4).

Strategies of B:

		B_1	B_2
Strategies of A:	A_1	(8, 5)	(10, 0)
	A_2	(6, 6)	(7, 0)

Figure 4. Bimatrix for choosing a solution.

In this positional game the team A makes decision already knowing the decision of the team B, which makes the first move. It is most probable that the team B can withdraw from the tournament due to many of its players are on the sick list and it is impossible to invite a good substitution (the team's rating is low in comparison to the team A). Regardless of the decision of the team B, it is advantageous for the team A to invite players for strengthening. Thus, in the positional game various ways of decision-making are clearly demonstrated.

Geometry is a science that studies graphical representations. It became widely popular already in the ancient times. The theorem that bears his name had been known long before Pythagoras (the VI-th century B.C.). At present the visual character of geometry is widely used, for example, in system analysis with using graph theory. Graph is a collection of vertices (nodes) and edges (arcs), which are a universal means of presenting a variety of tasks in a visual form. A typical problem is finding the shortest path (Глухов, 2000). Here it is required to find the shortest path on a graph from one vertex to another. For solution of this problem a method of discrete optimisation is applied. For example, assume that distances measured in hundreds of meters from a house (7) to other destination points and between them are known: 1 – the grocery store; 2 – the school; 3 – the market; 4 – the kindergarten; 5 –

the library; 6 – the rental centre. Let's map this in the form of a graph or structure (topology) of the network (fig. 5).

Nodes are the house (7), the school (2) and other points. Indexes of arcs (edges) are distances in hundreds of meters. It is required to find the shortest distances from the house to each of the points in order that a pupil would spend less time on the way to the school and other destinations, when it is necessary. This problem can be solved by a label-setting method. To each node we assign a label of two numbers: the first number is the minimal distance from the node 7 to the given node, the second one is the number of the previous node on the path from the node 7 to the given node. The node, for which we have defined the path from the node 7 is called a labelled node. If we have defined the shortest distance from the node 7 to the given node, then the corresponding label is called permanent and we write it in round brackets. Other labels are called temporary and are written in square brackets.

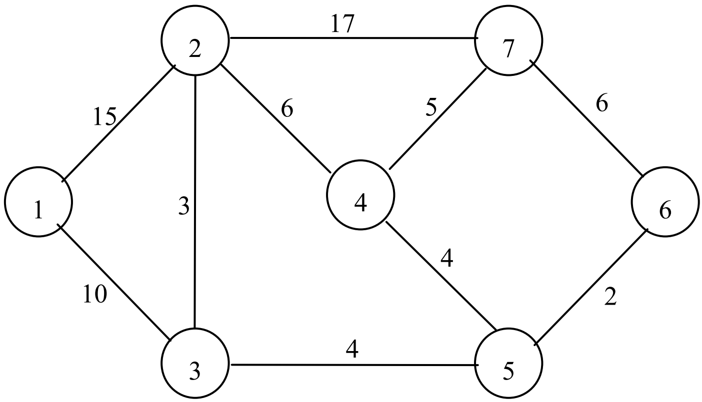

Figure 5. Structure (topology) of the network**.**

Thus, to the node 7 is assigned a label (0, S), where 0 is the distance from the node 7, S is a designation of the starting node. The node 7 is directly connected to nodes 2,4,6. The lengths of the corresponding edges are 17,5,6. Therefore to the nodes 2,4,6 are assigned temporary labels – [17,7], [5,7], [6,7].

After this operation it is possible to make two following steps:

to find a segment of minimal length and make the corresponding temporary label permanent;

the node, to which a new permanent label corresponds becomes a new start.

After such operation a temporary label with the minimal distance from the node 7 becomes permanent. This is a label (5,7) of the node 4. Then the next step is done from the node 4 (fig. 6).

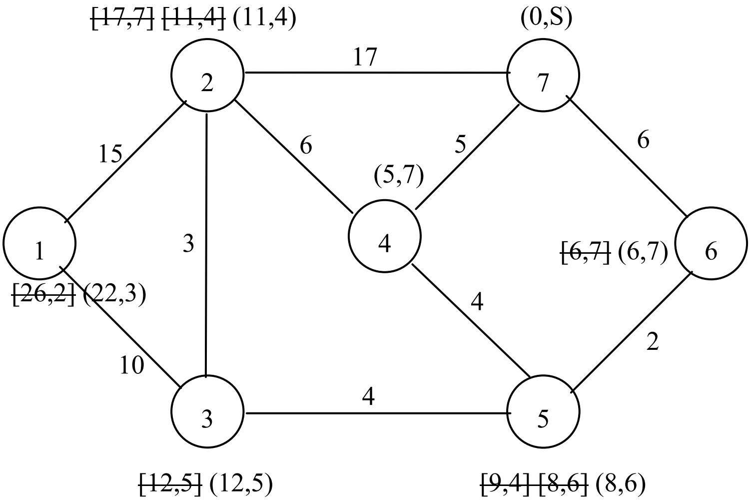

Figure 6. Giving nodes temporary and permanent labels**.**

The node 4 is directly connected to the nodes 2 and 5 without permanent labels. The length of the edge 4-5 is 4, then a temporary label of the node 5 is [5+4,4]=[9,4]. The length of the edge 4-2 is 6, a temporary label of the node 2 is [5+6,4]=[11,4]. The node 2 is marked with labels [17,7] and [11,4]. For the node 2 we choose a new temporary label with the shortest distance [11,4]. Thereafter from all temporary labels [11,4], [9,4], [6,7] we choose the label [6,7], where the first number is the least. This label becomes permanent and the next step we begin from the node, which corresponds to this label – 6. This node is connected to the node 5 by the edge of length 2. The node 5 already has a temporary label [9,4], however, the path from 7 to 5 via 6 is equal to 8 and it is shorter. Therefore we add a new label [8,6] to the node 5. Thereafter from all temporary labels – [11,4] and [8,6] we choose a label, where the first number is the least and make it permanent (8,6). We begin the next step from the corresponding node 5. The node 5 is connected only to one node without a permanent label – to the node 3. To the node 3 we assign a temporary label [8+4,5]=[12,5]. Now from all temporary labels – [11,4] and [12,5] we take a label, where the first number is the least [11,4] and make it permanent (11,4). We begin the next step from the corresponding node 2. The node 2 is connected to the nodes 1 and 3 without permanent labels. To the node 1 we assign a temporary label [11+15,2]=[26,2]. It would be possible to mark the node 3 with a label [11+3,2]=[14,2]. But the node 3 has a label, where the first number is lesser [12,5], therefore we do not change it. Now from temporary labels [26,2] and [12,5] a label, where the first number is the least, becomes permanent (12,5), and we begin the next step from the corresponding node 3. The label of the node 1 is changed to (12+10,3)=(22,3). Now all the nodes are given permanent labels and the algorithm stops.

The first number of the label at every vertex is the shortest path from the node 7 to the given vertex. In order to reconstruct the shortest path from the node 7 to any of the vertices, we have to move from this vertex to the neighbouring one (it's number is given by the second number of the label). And so on until the node 7 (fig. 7).

Now it is possible to answer the questions of the problem. For example, a label of the node 1 – (22,3): 22 is the length of the shortest path from the node 7 to the node 1. From the node 1 we move to the node 3, then to the node 5, then to the node 6 and, finally, to the node 7. This will be the shortest path: 1-3-5-6-7. The shortest path from the node 2 to the node 7, which is equal to 11, is via the node 4. This problem gives a visual representation of the simplest method of optimisation and the practical application of graph theory.

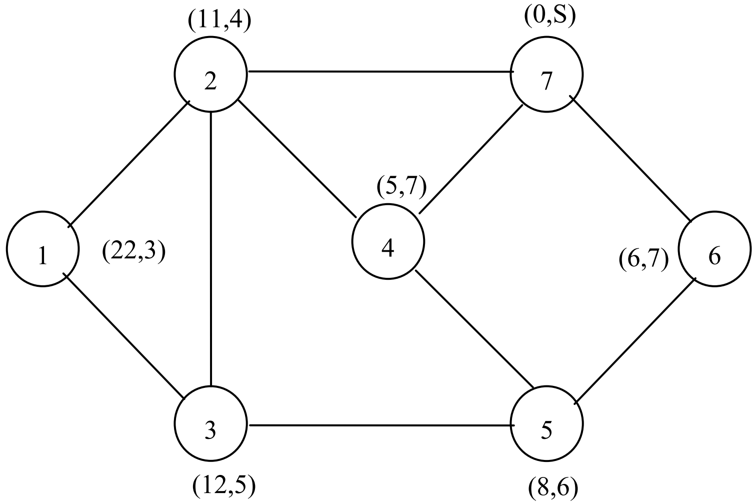

Figure 7. Topology of the network with permanent labels**.**

When studying mathematics and other relevant courses teachers usually use a traditional method "from the simple to the complicated". At first are given basic simple definitions, axioms and theorems. Thereupon are formulated more complicated definitions and concepts. In addition, very often first of all, as an illustration of theory, quite abstract examples are taken, but models, which can be used for studying real problems, do not find practical use. Therefore students often do not understand, what is the practical application of sometimes quite complicated mathematical theory, what for should they study it and where this material will be needed in the real life.

To eliminate this drawback it is possible, according to the author's experience, to use the methodology of applied system analysis. At first a preliminary study of a problem or an object and construction of a conceptual model is done. By the model is meant an image of a real object or a process, usually in the perfect form, which represents the most significant properties of the object (process) and substitutes it during the research. Then is performed mathematical formalization of the conceptual model and research thereof, interpretation of the results and the necessary iterations. Such sequence of material presentation has certain advantages. With practical models students observe the necessity of becoming familiar with new definitions and mathematical devices. There is a possibility to interpret mathematical concepts in terms of real prototype system. This helps provoke students' interest and develop their skills of practical application of the obtained knowledge.

As an example, structured modelling of complicated systems (Roberts, 1986) can be considered. The structured model of a complicated system is a signed oriented (directed) graph (digraph): vertices of the digraph correspond to the elements of the system; an arc from a vertex u_i to a vertex u_j is constructed only when the change of u_i value has a significant influence on the change of u_j. To the arc (u_i,u_j) is assigned the "plus" sign if u_j increases with the increase of u_i, and u_j decreases with the decrease of u_i; the "minus" sign is assigned if u_j decreases with the increase of u_i and vice versa. The use of signed digraphs gives a possibility to make quality conclusions about the modelled system. Contours and semi-contours in a digraph correspond to certain feed-backs: negative in case of reaction against deviations, and positive in case of amplification of deviations. Such representation of a complicated system with the help of signed digraph gives an opportunity to display the whole collection of main system elements as well as the presence and direction of connections between the elements. If it is possible and necessary, instead of a signed digraph it is possible to use a weighted digraph, assigning to an arc not only signs, but certain numerical values, for example, probabilities.

The use of graph theory for formulation, analysis and solution of practical problems, which are quite simple at the first stage, creates a vivid interest in students, enhances development of systemic thinking, and teaches them to define the main factors from the great numbers existing in real-life situations.

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STUDENTS` UNDERSTANDINGS OF RATIONAL NUMBER REPRESENTATIONS

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Abstract

A full understanding of rational numbers for any middle grade student involves an understanding of rational numbers in all of its representations and sstudent's ability to operate effectively with rational numbers depends on knowing interconnections among different representations of rational numbers (numeral and pictorial). The paper will present some preliminary results of the study on students' understandings of the connections among rational number different representation forms.

Teachers spend a large number of lessons to teach their students arithmetical shifts with common and decimal fractions. Another time-consuming enterprise is to teach students which operations are necessary to solve certain types of tasks with percentages. Unfortunately, teaching the concepts and connections regarding rational numbers is often limited to cramming large amounts of rules and algorithms. This may result in a rather superficial understanding of the concept of rational number, its different meanings and representations (Stohl , 2002).

Therefore one should not be later taken by surprise if, for example numbers $\frac{1}{7}$ and 0,7 are considered to be equal, 0 is offered as a solution for subtraction $\frac{5}{6} - 5{,}6$. Another misconception is that a common fraction is always smaller than 1, whereas only some certain object can be whole as opposed to fractions. Researches dealing with problems relating to teaching rational figures reveal that for example understanding the rational number $\frac{4}{5}$ presumes the ability to present it as a decimal fraction, percentage or another common fraction (0,8;

80%; $\frac{12}{15}$). At the same time researches have stressed that students should recognise and be able to operate also with pictorial representations of rational numbers. It is important to mention that visualising a rational number one must use as many different objects or amounts of objects as possible (see also: Figure 4) as playing with different constructions supports the development of multiplicative reasoning and gives a deeper understanding of such meaning of rational number as ratio. (Kurvits, 2008; Lamon, 1999)

The paper will present some preliminary results of the study on student's understandings of the connections among rational number different representation forms. This study was conducted in one of the comprehensive schools of Tallinn, Estonia and it is a part of longitudinal study on rational number and multiplicative reasoning, which began in autumn 2007. During the school year 2008/2009 were tested 76 6th graders (three parallel classes), the same students were tested during the school year 2007/2008 and according to plan they will be tested during the next school year 2009/2010.

In this article we pay more attention to the pictorial representations of common fraction. Let us start with the first task of the test I, where the fraction $\frac{3}{4}$ is asked to be presented in some other way (Figure 1). All in all the percentage of incorrect answers was 49 and a quarter of those suggested $\frac{4}{3}$ as a correct answer. Another interesting fact was that 14 percent of the students wrote their answers in words: three quarters, and not a single student was able to present the fraction graphically.

In second test numbers $\frac{1}{4}$ and $2\frac{1}{4}$ were asked to be presented graphically. Figure 2 gives an overview of the corresponding results.

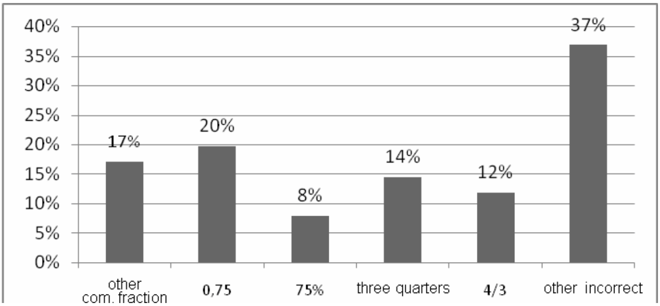

Figure 1: Representations of fraction $\frac{3}{4}$

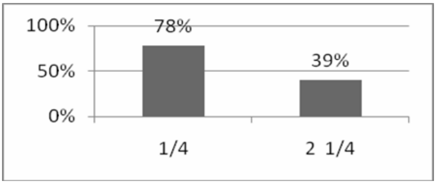

Figure 2: Representations of $\frac{1}{4}$ and $2\frac{1}{4}$

Only 39% of the students could offer the right drawing for presenting $2\frac{1}{4}$, whereas Figure 3 depicts the "popular" incorrect option for presenting $2\frac{1}{4}$.

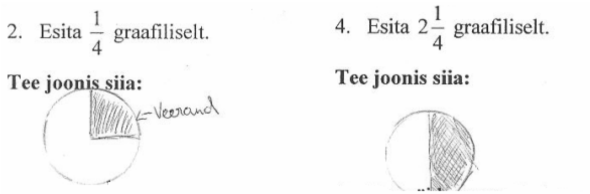

Figure 3: "Popular" incorrect representation of $2\frac{1}{4}$

In Test IV charts 1-22 were the tasks where fraction $\frac{3}{4}$ was asked to be presented (Figure 4). Figure 5 gives the percentage of correct answers. Charts 1, 2, 9 and 17 produced the best results. That was not surprising as most teachers use these constructions when introducing common fractions. Regrettably they often limit their explanation with this pictorial representation.

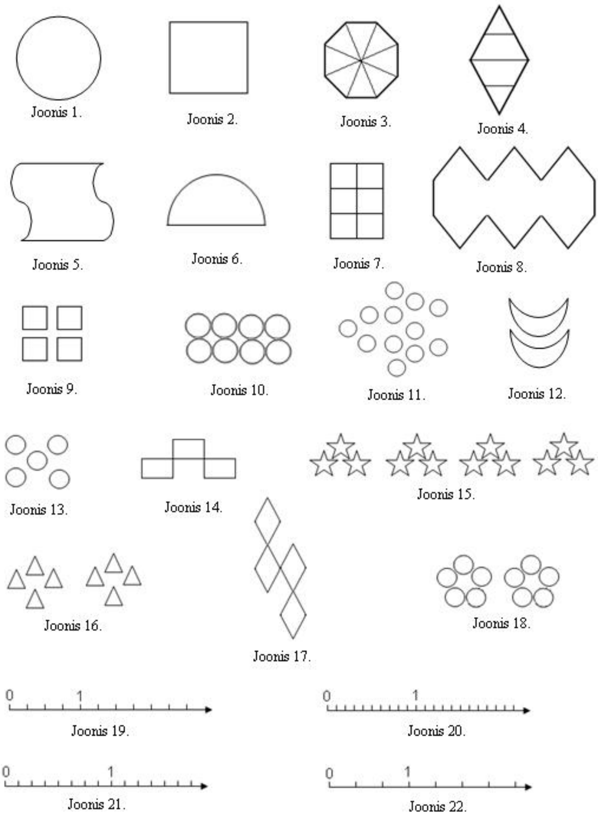

Figure 4: Charts for representing $\frac{3}{4}$

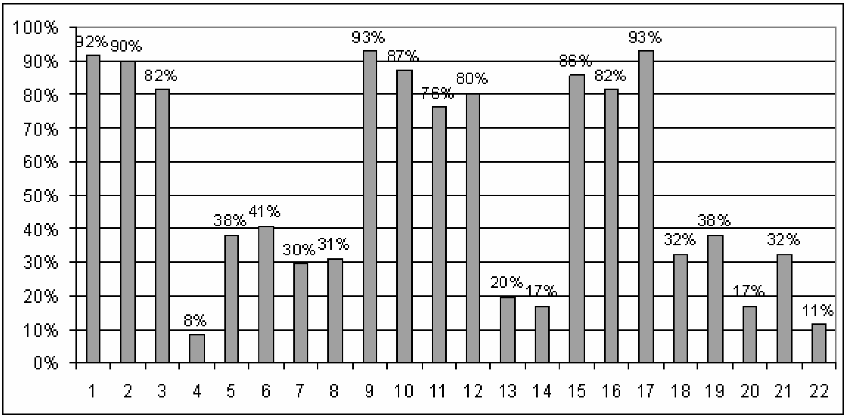

Figure 5: Results of representing $\frac{3}{4}$ in Test IV

Chart 4 turned out to be the most difficult one. Only 8% of the students picked the correct option for presenting $\frac{3}{4}$. The majority of students did not notice that the whole was not divided into three equal parts.

The results for Charts 13 and 14 were 20% and 17%, respectively. Many students suggested the chart where one part was left uncoloured (Figure 6).

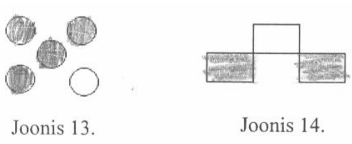

Figure 6: Incorrect responses for representing $\frac{3}{4}$

The reason for this was that a number of students think that $\frac{3}{4} = \frac{4}{5}$ or that $\frac{3}{4} = \frac{2}{3}$ (as both fractions have "one bit missing" or, in other words only additive connections between the numerator and the denominator are noticed). It is important to mention that In Test VII the students had to compare different rational numbers and 26% of them thought that $\frac{3}{4}$ equals to $\frac{5}{6}$ (in both fractions the numerator is bigger than the denominator by one).

The results regarding Chart 19 were surprising (38% correct answers). Many students ignored number 1 on the number scale and offered the option depicted in Figure 7.

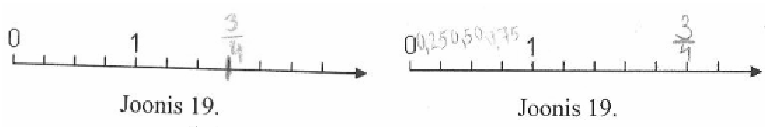

Figure 7: Incorrect responses for representing $\frac{3}{4}$

The difficulty of charts 5, 6 and 8 (percentage of correct answers 38, 41 and 31, respectively) lay in the fact that these images "did not look common" and therefore the division of the whole into parts caused a lot of mistakes (Figure 8). However, 17 percent of the students suggested the construction depicted below to be the solution for Chart 8 (another case of unjustified use of additive connections).

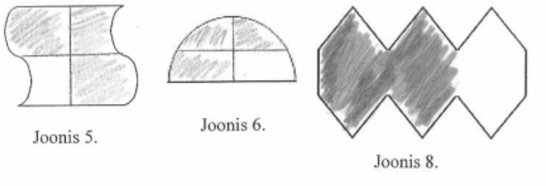

Figure 8: Incorrect responses for representing $\frac{3}{4}$

The score of Chart VII was only 30%. The main reason for this was the fact that initially the rectangle was divided into six parts and six cannot be divided by four. Many students then altered the chart in such manner that presenting fraction $\frac{3}{4}$ would be graphically easier (Figure 9). In fact this strategy was implemented also on other charts (Figure 10).

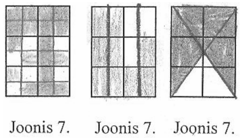

Figure 9: Correct responses for representing $\frac{3}{4}$

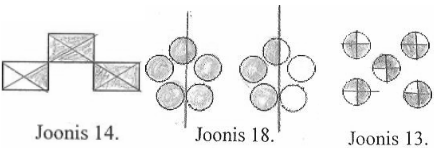

Figure 10: Correct responses for representing $\frac{3}{4}$

It might be of interest to return to the examples presented in the very first paragraph as the described tests contained the tasks ("compare $\frac{1}{7}$ and 0,7", as well as "subtract 5,6 from $\frac{5}{6}$").

Sadly, 47 percent of the students offered that $\frac{1}{7}$ = 0,7 and 33 percent suggested that $\frac{5}{6} - 5{,}6 = 0$. It is likely that the reason for such mistakes is the fact that algorithms acquired by cramming and rules without a meaning attached to them are remembered for a very short period by students. In a few months the teacher then is forced to revise the material thoroughly so that the students would be able to pass standard tests. Regrettably, the students' ability to memorise things is getting poorer by the year.

Consequently, while teaching rational numbers a teacher should pay special attention to the variety of their presentation (both numeral and pictorial) as it results in students' deeper understanding of and efficient operation with the concept.

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BLOG IN SCHOOL MATHEMATICS

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Marina Kurvits, Tallinn High School of Technology

Abstract

Most teachers know what a blog is and how to create one, but they find it hard to see how it could be used to educate. In this paper we will share our experience of using a blog to teach mathematics and explain why we decided to create one. We will also give examples of additional web-tools and services which were used to increase interactivity of the blog.

A large number of people have, at least to some extent, had experienced with a weblog or blog. In principle a weblog is a website where an individual can add entries. Entries are commonly presented in a reverse-chronological order and they may contain presentations, drawings or for example, videos. A weblog can also be maintained by several individuals and it is also possible to comment on the posts. A weblog can either be available for the general public or a certain community. Millions of people have created their own blogs and blogging has gradually arrived at Estonian schools where weblogs are used for several purposes.

As for us the first idea to create a weblog sprang up after a conference "E-study is already here, where are you?" held at Estonian IT College. As Marina works as an information manager in Tallinn High School of Technology she decided to create a weblog, a purpose of which would be the information exchange within the school. At the same time we have often experienced that communicating with our students at lessons only is not sufficient. Therefore we have used the Internet as an additional means of communication. We have uploaded various tasks, worksheets and materials for revision, whereas feedback has taken place via e-mail. We felt that we were able to offer our students a lot more than we managed to do merely at the lesson or during our consulting hours. While talking to our students we realised that very few of them have a chance to turn to anyone for help in the case of troubles relating to maths. This situation often results in homework not done as soon as the first problems crop up. This is the main reason why we decided to create a fully interactive maths teaching weblog with an active participation by the students. As a great help to do so we acquired a lot of positive emotion and exciting ideas at a conference "From Teacher to Teacher" held at the end of October 2008 at Tallinn Secondary Science School. This is how the weblog titled "All Maths Fans Unite!" came into being (Figure 1, Figure 2.).

As due to the students' wish the particular weblog is not available for the general public we would like to give an overview of its structure. The left-hand area contains posts on different matters including the information about homework, tips for tests, various presentations, problems solved by ourselves, training videos, etc. The right-hand area comprises various study materials, useful links, chat widget, links to online whiteboards (*Dabbleboard*), online documents (*Google Documents*) and online file storages (*WebAsyst*), two calculators the graphic calculator of which enables to draw graphs of functions directly on the weblog.

In the section above one can find the archives of the blog and the form for posing questions. Somewhat later we also added a feedback form via which the students have expressed their opinion about our blog and proposed ideas for its improvement. As soon as a student via the designated feedback form has expressed that he or she has trouble with some problem, the teacher involved gets a corresponding message to his or her personal e-mail. As our personal mailbox is almost constantly open we are able to react to our students' queries in no time. But from our experience we can say that a lot of students tend to be active at very late hours when the teacher is going/has gone to bed.

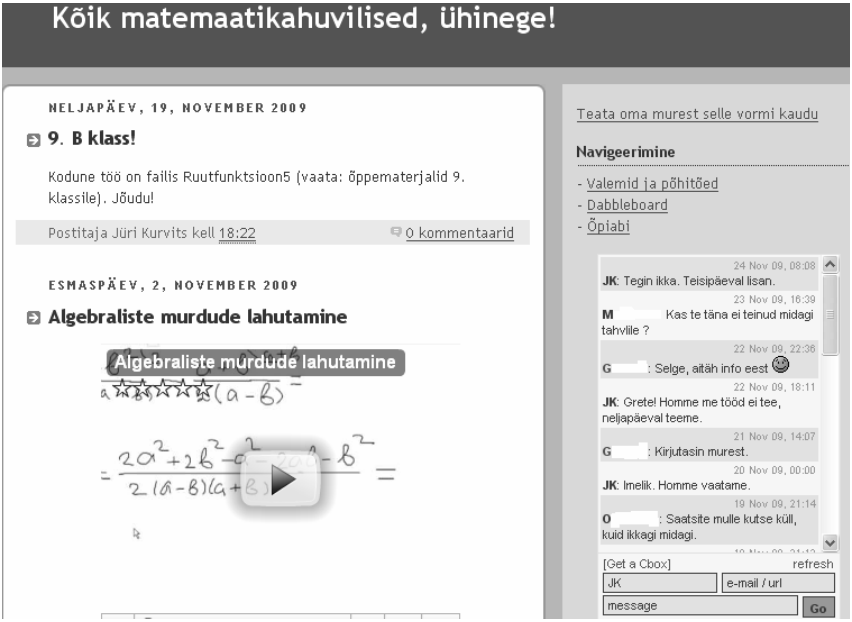

Figure 1

Figure 2 shows that the right-hand section includes the archive of files. At present there are five of them: Practical Mathematics, Files with Problems from Students, File Record: Form 11, File Record: Form 12 and File Record: Form 9. As our classrooms is equipped with an interactive SMART-board which enables to save all the texts and drawings in the form of images, we have after each lesson added a file to the archive to record all the tasks performed. This has been appreciated by the students who, for some reason or other have missed a class and who, thanks to this opportunity are able to familiarise themselves with the jobs done. On top of all this it is also possible to ask the teacher additional questions using the designated queries' form.

Students have also an opportunity to make their own entries into the files (archive of files: Files with Problems from Students). Should a problem arise with solving a problem a student can always add an image of the page of their notebook containing the troublesome problem as a separate file to the archive, which contains an alternative solution to the particular problem. When we open the file we can provide the students with useful tips or add another file containing an alternative solution to the problem.

Using these archives of files we have conducted tests which the students can take from home. The arrangement is as follows: at a fixed time the tasks appear in the weblog and by the next morning a corresponding file with solutions by the student is expected for the teacher to appear.

Figure 2

Any student has an opportunity to comment on the posts and sometimes vivid discussion can arise (up to 30 – 40 comments to a single post). A positive aspect of such exchange of comments is that through the discussion the students may find a solution to the tough problem on their own, without the teacher interfering them. The teacher, though, instantly receives an e-mail about a post by a student.

The school-leavers thus should presumably have a substantial experience of team-work and they are expected to have skills in both oral and written expression of their ideas. We have been trying to provoke our students to express themselves and their ideas at the lessons designed for such response. We often introduce a new topic avoiding to provide the students with already existing solutions or reading theorems to them. Instead we try to create a situation where students can reach the solution "on their own". In this the materials added to the weblog have proved to be useful as these are easy to demonstrate by using the projector. While explaining new material we try to avoid any "pre-digested" solutions. A maths teacher cannot be just a lecturer and let the students be passive scribes. Maths does mean active brainwork. This consequently means an active dialogue between the student and the teacher. We as the teachers felt an urge to continue this dialogue after the formal lesson had finished.

We have always found it of ultimate importance that questions are asked at a lesson as, in our opinion, studying any subject without asking any questions seems impossible to us. But students are very different and some of them never ask questions in the class. In this situation our weblog offers help as it is possible for a student to turn to us individually or discuss problems with peers. From our own experience we can say that these students who seldom utter a word in the class during the whole week may be quite active commentators on-line. A reason may be the fact the web-based environment is natural and familiar to present-day students. It is easier for a student to open the weblog than to find a textbook on the bookshelf. It is also worth mentioning that not a single student has doubted whether to join the weblog. In fact we did not bother ourselves too much about what they had to do in order to join in, we just sent an invitation on each student's e-mail and they did the rest independently.

Having noticed the students' interest we decided to create one more option for interactive communication: files in Google documents environment, one for ninth-formers, one for tenth-formers and the other for twelfth-formers. These files are called Spreadsheet and they are very similar in shape to the Microsoft Excel files (Figure 3.).

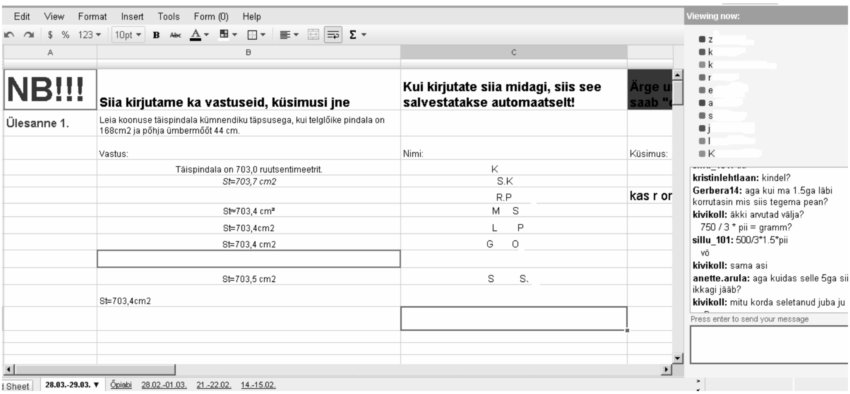

Figure 3

These files are on-line documents and they include tasks added by us. It is possible to write questions, answers, solutions to the tasks, etc. in the same file. Especially noteworthy is the possibility of chatting (in figure 3 on the right-hand strip it can be seen that on the moment of taking this photo there were ten people working on this file). In fact we could not believe that the student would be able to collect that much material during a chat one night.

We have used the previously mentioned files for weekend tasks and some colleagues are surprised that students have starting to ask for additional homework.

In fact we have also used the Google documents environment also for tests, quizzes/polls, presentations, preparation of study materials. All study materials on our weblog are in Google documents environment. Google documents are constantly on-line and using them noticeably simplifies interactive group work. Users who have been given access to these documents by their authors can read or change them. Any changes can be seen immediately. This feature makes the use of these documents convenient for organising work in groups at the school's computer class.

We have received a lot of positive comments as feedback. Students are satisfied and they hope that other teachers also start creating their own weblogs. Students were especially impressed by the fact that the only prerequisite to make use of the materials was connection to the Internet. Our students have pointed out that Google documents are ideal media for transmitting homework, as during the process of solving the problems they can discuss matters with other students or turn to the teacher. This has proved to be useful while preparing for tests. We should certainly say thank you to our students for their active and enthusiastic attitude towards the new study environment. We sincerely hope that our reader gets inspired to try something similar. We are certain that in the case of interest technical support should be available.

TECHNOLOGICAL AIDS OF THE LEARNING CYCLE IN A BLENDED TEACHING OF MATHEMATICS

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Antanas Lipeika, Vilnius Pedagogical University

Abstract

The paper discusses the use of technological tools in the learning cycle scheme of blended teaching of mathematics at the university level. Blended teaching is understood as a combination of teaching face-to-face and on-line. Authors describe a possible content of courseware that could be used at the conceptualization, construction and dialogue phases of a learning cycle.

Introduction

Learning theories have fundamental philosophical differences, however, nowadays educational designers tend to think that a combination of learning theories works best (Hadjerrouit, 2008). Mayes and Fowler (Mayes and Fowler, 1999) proposed a three stage model or a learning cycle of blended learning involving three types of learning – conceptualism, construction and dialogue (Fig. 1).

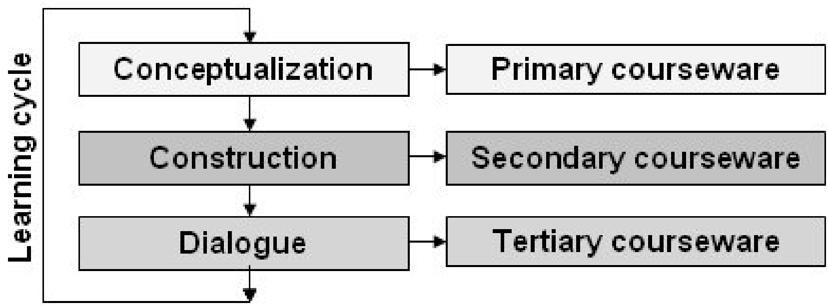

Figure 1: Mayes and Fowler's learning cycle scheme.

Conceptualism is associated with the *cognitive learning* as it focuses on concepts and their relationships. The construction stage aims to construct new knowledge by solving practical tasks so it is related to the *constructivism*. And the dialogue concerned with dialogue and collaboration is based on the *social constructivism* theory. The latter theory sees learning as a participation in social processes of knowledge construction. The authors (Mayes & Fowler, 1999) have also characterized three types of information technologies (IT) used to achieve each stage of the learning cycle: *primary* courseware (educational software) mainly intends to present concepts of the subject matter, *secondary* courseware supports the performance of task-based activities, and *tertiary* courseware that consists of dialogues between learners and teachers, presents online discussions and collaborations (Fig. 1).

The present paper discusses the use of Mayes and Fowler's scheme for blended teaching of mathematics at the university level, describes applicable for mathematics technological tools and presents a possible content of the courseware of the each stage. Blended teaching here is understood as a mix of teaching and learning face-to-face and on-line. The understanding of technology role in each stage of mathematics teaching is very important. It explains possible misunderstandings and doubts about the usefulness of technology in mathematics teaching. What are the contemporary technological tools applicable at each stage of the learning cycle? What tools could comprise the primary, secondary and tertiary courseware in order to be successful in teaching? The paper presents answers to the questions and some discussions on the technology use at each stage.

Information technology tools for mathematics

Let us look what applicable for mathematics information technology (IT) tools can comprise the each courseware of Fig. 1. One can include to the primary courseware, intended for conceptualization phase, first of all, visualization tools of computer algebra systems (CAS), other educational programs, and also various mathematical resources on-line. Construction phase, the secondary courseware, should include visualization tools of computer algebra systems, mathematical resources of Internet, special procedures of CAS and also mathematical games. In the dialogue phase of the learning cycle, the tertiary courseware should consist of mathematical virtual learning environments, also teacher's blogs, if such are available for the subject (Fig. 2).

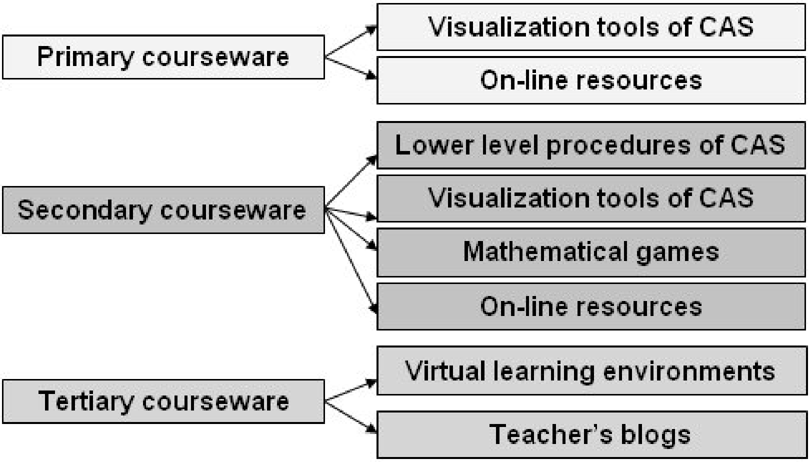

Figure 2: Technological tools comprising primary, secondary and tertiary courseware.

Conceptualization

The authors earlier investigated IT visualization tools of CAS (Lipeikiene, 2005; Lipeikiene and Lipeika, 2006; Lipeikiene, 2008). At the conceptualization phase, teachers focus on understanding of concepts and their relationships. 2D and 3D graphics, animation facilities of CAS help to increase understanding. CAS have various graphic facilities to describe mathematical concepts and explain relationships (Fig. 3). Also a possibility to demonstrate the correspondence between algebraic and graphic representations is widely used.

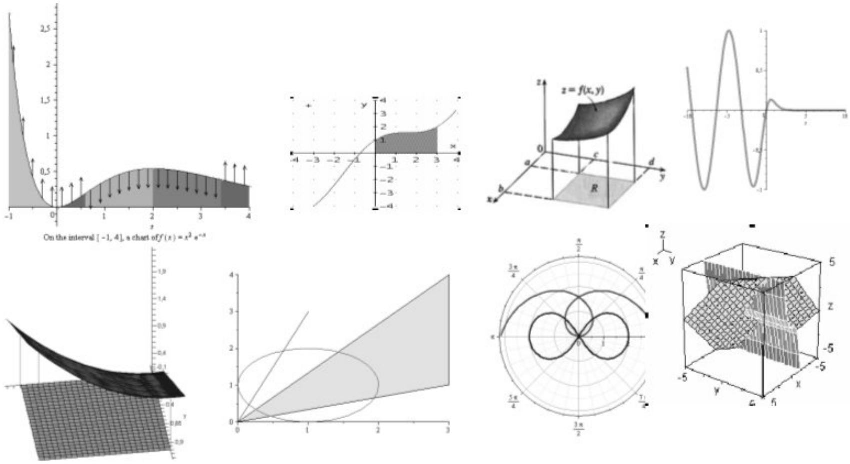

Figure 3: Various useful for conceptualization graphic facilities of CAS.

In addition, many functions of CAS are specially designed to explain concepts or visualize concepts. For example, Maple functions leftbox, rightbox, LimitPoint, NiutonsMethod, TaylorApproximation, CompositionPlot, DerivativePlot and so on, provide not only description but also an explanation of the concepts. Animation is a special visualization tool for conceptualization phase. It offers opportunities for visualization of complex mathematical concepts, illustrates ideas and influence of quantities or parameters, helps to generate hypothesis, encourages exploration. Animations can be used to demonstrate many mathematical concepts that are difficult to explain verbally or to show with static pictures. We should include also mathematical Internet resources that could be useful at the conceptualization phase. Educypedia (http://www.educypedia.be/), the best of the Web for maths (http://www.vts.intute.ac.uk/ he/tutorial/maths), and a catalog of Maths resources (http://mthwww.uwc.edu/wwwmahes/files/math01.htm) gives a general view of the resources. One can summarise all applicable at the conceptualization phase tools in Fig. 4.

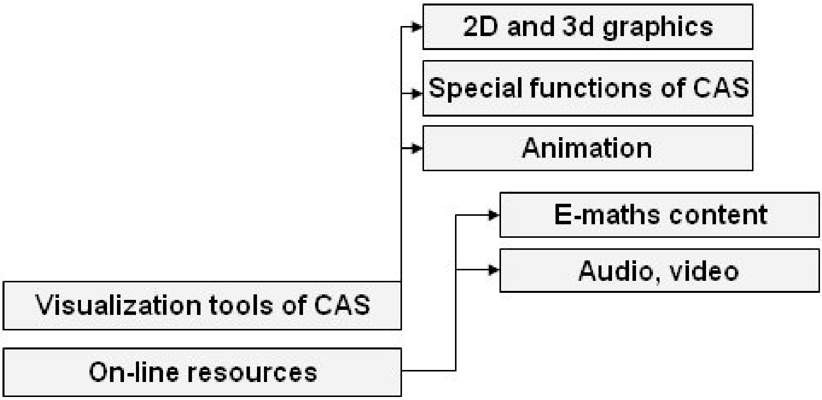

Figure 4: The scheme of technological tools applicable at the conceptualization phase.

Construction

The secondary courseware, intended for construction phase, is devoted to facilitate task-based activities, mostly self-dependent laboratory work or homework. First of all, lower level procedures of CAS, such as functions of simplifications, transformations and calculations (simplify, collect, combine, convert, and so on), should help in solving various problems. They facilitate boring calculations and free from simple but tedious work. Evaluation functions help in evaluation of answers with desirable precision. The notion "lower level procedure" is versatile. It depends on a subject taught. For example, when teaching integration simplification procedures and graphics are useful procedures of lower level, but functions of integration are procedures of the same level, which should not be used. But when students already know integration and solve problems with application of integrals, they should use integration procedures as procedures of lower level. Of course, the possibilities of visualization and internet resources, already discussed in the previous section, are also applicable at the construction phase. Therefore tools of construction phase consist of all items, enumerated in Fig. 2 as Secondary courseware.

Dialogue

The dialogue phase of the blended teaching is devoted to collaboration between students and teacher, to active students' participation in the learning process, and to testing students' understanding. This phase can be performed separately or in parallel with the first and second phase. The phase includes dialogue face-to-face and dialogue on-line. The latter could be realized using virtual learning environments (VLE) (Lipeikiene, 2003) or teacher's blogs. Both have to be created by teacher. The creation and using of VLE always require much teacher's efforts and time.

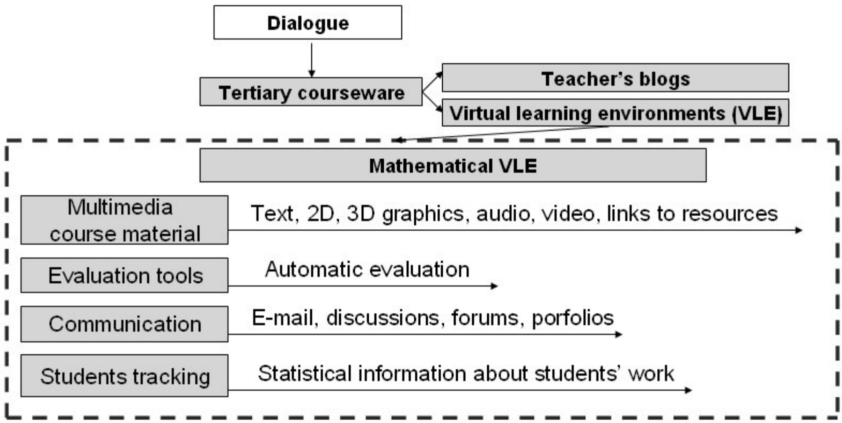

Figure 5: The scheme of a dialogue phase.

Blackboard Vista (commercial) and Moodle (Open Source) are the most popular tools for creation of VLE. Fig 5 illustrates the content of on-line dialogue phase.

Conclusions

The investigation of applicable for mathematics teaching technological tools shows how versatile IT tools are. They could be used in all three phases of blended teaching of mathematics according to the presented in the article schemes of each phase. Primary courseware, intended for conceptualization, includes visualization tools of CAS and Internet resources. Secondary courseware should comprise of visualization tools of CAS, Internet resources, lower level procedures of CAS, and mathematical games. Finally, tertiary courseware should be virtual learning environments, created for mathematical subjects. All tools of the each phase of teaching can be used separately or in parallel with tools of other phases.

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PROPORTION IN SWEDISH UPPER SECONDARY SCHOOL TEXTBOOK TASKS

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Abstract

This study investigates what possibilities Swedish upper secondary school textbook tasks offer students to develop their understanding of proportion by applying an analytical tool on one textbook. We found a variety of tasks, but some proportional reasoning types were missing.

Introduction

Proportional reasoning, understanding of ratio, proportion and proportionality are prerequisites for success in higher studies in mathematics and science but also for many practical applications. Research shows that these topics are difficult even for many adults (e.g. Keranto, 2004). We have noticed that Swedish upper secondary school students still have problems, for example when they change units of velocity. They very often use a trial and error strategy. Most of the studies conducted in the field shed light on the teaching and learning of these notions in primary and lower secondary grades but we have very little knowledge about what happens during the upper secondary school (Lamon, 2007).

The purpose of this study is to investigate what possibilities Swedish upper secondary school textbook tasks offer students to develop their understanding of proportion and proportionality during the first course in mathematics. The first year upper secondary school level is chosen as it is the base for all the further studies for both the theoretical and the practical programs at upper secondary school.

The notion of proportion in textbooks

According to Lamon (2007) students' proportional reasoning develops when they build up rational number sense through varied experiences. Proportionality refers to a condition in which a special invariant relationship exists between two co-varying quantities. We take the view that tasks involving proportional reasoning deal with interaction linking two rational expressions for example ratios, rates, quotients and fractions (Lesh, Post, & Behr, 1988).

We consider textbooks as important artefacts in the teaching of mathematics. Research shows that mathematics textbooks have a dominating role in the teaching of mathematics in Sweden as teachers usually choose examples and tasks from the textbook (e.g. Johansson, 2006). That is why it is necessary to study what kind of exercises and possibilities to insights about proportion and proportionality the textbooks offer to the students. However, students' encounters with the mathematical tasks are unique and situated and the analysis in this study just attempts to characterise the tasks from various points of view in order to discuss what *possibilities* for learning they offer.

Methodology

We chose *Matematik 4000 kurs A blå* (Alfredsson, Brolin, Erixon, Heikne & Ristamäki, 2008) because it is the most commonly used textbook in upper secondary school classes in our region (three municipals). The textbook is newly written and has been on the market for one year and builds on an earlier well known textbook series.

We used a technique of content analysis in the following way. Firstly, we analysed the tasks according to the cognitive demands that are demanded for solving the tasks. The PISA-framework (OECD, 2003) divides the cognitive demands into three clusters: *reproduction*, *connection* and *reflection*. The reproduction cluster involves reproducing practiced materiel and performing routine operations. Example of a reproduction task:

> "Write 69% as a fraction." (OECD, 2003, p. 43).

The connection cluster has its grounds in the reproduction cluster but the situations in these tasks are not simply routine but still within a well known setting. The keywords for the connection cluster could be described as connecting, integrating and modest extension of practiced material. Example of a connection task:

> "Mary lives two kilometres from school, Martin five. How far do Mary and Martin live from each other?" (OECD, 2003, p. 45).

The reflection cluster includes elements of reflectiveness and planning of the solving process. The settings of the problem should be unfamiliar too. Example of a reflection task:

> "Now Susan wants to make a block that looks like a solid block that is 6 small cubes long, 5 small cubes wide and 4 small cubes high. She wants to use the smallest number of cubes possible, by leaving the largest possible hollow space inside the block. What is the minimum number of cubes Susan will need to make this block? (OECD, 2003, p. 80).

All these three clusters were used in our data analysis.

In order to enhance students' interest and engagement in mathematics it is necessary to offer a variety of situations for using and doing mathematics (Marton & Booth, 1997). That is why we also apply PISA's assessment component concerning the context of the tasks. The context of the tasks can be, based on the PISA framework, *intra mathematical*, *personal*, *educational/occupational*, *public* and *scientific*. See table 1. The PISA framework classifies the intra mathematical tasks into the category of scientific tasks whereas we, in line with da Ponte & Marques (2007), distinguish intra mathematical tasks from application tasks.

Intra mathematical	A task that only refers to mathematical objects, symbols or structures
Personal	Involves personal situations
Educational/ Occupational	School situations activities / Professional activities for the student in the future.
Public	Life in society, community
Scientific	Tasks from science areas.

Table 1: Description over the context categories.

Giving students arming with a variety of perspectives and solution strategies sustain not only better understanding but also a more convinced and flexible appeal to problem solving that is why we also studied what kind of proportional reasoning and proportionality was involved in the tasks: missing value, numerical comparison, and qualitative prediction & comparison (Lesh, Post, & Behr, 1988). Missing value, tasks are where three pieces of information are given and the task is to find the fourth or missing piece of information. Numerical comparison tasks are where two complete rates/ratios are given and a numerical answer is not acquired, but the rates or ratios are to be compared. Qualitative prediction and comparison are tasks that require comparisons which not depend on specific numerical values.

The categories above are developed for the lower grades, but in upper secondary school proportion with graphs is also needed, so a new category is developed *decide k.* This is a variant of missing value tasks where the task is to decide the proportional constant out of one pair.

Lesh et al tasks only involves direct proportion whereas we include also other types of proportionality in our study like *inverse proportion*, *square proportion* and *square root proportion.*

Finally, we analysed the openness of the tasks. Open tasks require a more extended response from the student and involves high order cognitive activities (OECD, 2003; Taflin, 2007). We judge a task to be open if the solver constructs the formulation of the task or if the theoretical answer depends on the solver (Becker & Selter, 1996). In closed tasks, the mathematical givens, goals and conditions are explicitly expressed.

Hence we use the following categories in our data analysis: Cognitive demand: reproduction, connection and reflection.
Context: intra mathematical, personal, educational/occupational, public and scientific.
Proportional reasoning: missing value, numerical comparison, qualitative prediction & comparison, decide k and other.
Proportion type: direct, inverse, square and square root proportion.
The reliability of the analytical tool can be tested by a research fellow using the tool on the textbook tasks. This was done in this study.

Results

The book has five chapters and in the chapter about functions there is a section called *graphs & proportions* that specifically deals with proportion and proportionality. There is a short introduction to the topic in the beginning of the section with some examples of tasks and the solution of them. The introduction is followed by 43 tasks. The tasks are classified into three levels of difficulty (A-C) by the text book authors. The section about graphs and proportions are concluded by some extra materiel ("Activity") for the students to work with to develop their skills of problem solving. All the 43 tasks in the section were analysed.

Concerning the cognitive demand, the reproduction tasks were most common (61%), 25 % were connection tasks and 14% were reflection tasks.

Example of a connection task:

> "4138 Rewrite the formula and decide if y is proportional to x. If so decide the proportional constant. $\frac{x}{3} - \frac{y}{9} = 0$ "(Alfredsson et al, 2008, p. 204) Auth transl.

Almost every task (92 %) in the reproduction cluster was categorized as the easiest tasks (A-level) by the text book authors. 83 % of the tasks in the connection cluster were B-level tasks whereas half of the tasks in the reflection cluster were B-level tasks and half of them C-level tasks.

Concerning the context of the tasks, intra mathematical tasks were the largest represented category (47%). Public context was the second with 32% closely followed with scientific context (21%). No tasks were found in the categories personal and educational/occupational.

Only four tasks concerning proportion type were found to be missing value tasks and the numerical prediction category was represented by three tasks. There were no tasks representing qualitative comparison and prediction. Decide k was represented in seventeen tasks and the rest was categorised as other (18 tasks).

Example of a missing value task:

> "4135 The resistance of a thread of copper is direct proportional to the length of the thread. A thread of 1,25m have the resistance of 10,5 ohm. How long thread gives the resistance of 25,0 ohm?" (Alfredsson et al, 2008, p. 204) Auth transl.

Most of the tasks (54 %) involved direct proportionality but all the following types of proportionality were represented. See table 2.

Proportion	Direct	Square	Inverse	Square root
Formula	$y = k \cdot x$	$y = k \cdot x^2$	$y = \frac{k}{x}$	$y = \frac{k}{\sqrt{x}}$
In %	53%	20%	20%	7%

Table 2: Overview of proportion types found in analysed textbook.

None of the tasks were open tasks either in the regular section or in the activity section. The reliability of the analytical tool showed differences in the categories connection and open.

Conclusion

The reliability of the analytic tool was satisfying apart from the categories connection and open. These categories have to be better defined for future analysing. All three cognitive demands were represented but we saw very little of the reflection tasks. We can observe that the classification of the tasks into three levels of difficulty (A-C) matched quite well with the three categories of cognitive demand. Concerning the context of the tasks, there were very few scientific tasks. They are important as they help students to see the connections between mathematics and science and earlier research shows that they develop a deeper understanding for proportionality (Lamon, 2007). In Sweden, the role of open and rich problems has been in focus in the field of mathematics education (e.g. Taflin, 2007). Hence, it is remarkable that, none of the proportion tasks in the textbook, not even in the Activity section were open which could be useful for problem solving. However, variations of tasks offer opportunities for learning so it would be recommendable to offer more kinds of proportional reasoning types of tasks.

The next step in our study is to investigate how the variety of the tasks in the textbook mach to tasks in the national examinations. We will also study students' encounters with these tasks by analysing their solutions.

Acknowledgement: This study is funded by The Bank of Sweden Tercentenary Foundation and The Swedish National Graduate School in Mathematics, Science and Technology Education.

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AN ELEMENTARY PROOF OF NUMBER PI IRRATIONALITY

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Abstract

Elementary methods of the number theory are of particular interest. No less important is attention to simple proofs of irrationality of known constants and values of analytical functions. A brief school proof is known that all the values (except for some trivial cases) of trigonometric functions of the arguments $m\pi/n,\ m,n \in \boldsymbol{N}$, *are irrational. The school proof of the irrationality of number e has already been known for ten years, while such proof for number π succeeded to be devised only several months ago. The method presented requires neither infinite series nor integral theory.*

The widely known proof of irrationality of number π belongs to Niven (1956). Another method needs the Taylor series and integrating (Mačys, 2008).

A brief proof of the irrationality of π

Lets assume on the contrary that π is rational, $\pi = p/q = 2p/(2q) = m/(2q)$. Then $\cos 2q\pi = \cos m = 1,\ \sin m = 0$.

Multiply the Taylor series for $\sin x$ by *x* and integrate:

$$\int_0^x t\sin t dt = -x\cos x + \sin x = \frac{x^3}{3} - \frac{x^5}{5\cdot 3^5} + \frac{x^7}{7\cdot 5} - \dots .$$

Multiply the latter by *x* and integrate once again:

$$-x^2\sin x - 3x\cos x + 3\sin x = \frac{x^5}{5\cdot 3} - \frac{x^7}{7\cdot 5\cdot 3} + \frac{x^9}{9\cdot 7\cdot 5} - \dots .$$

Obviously, multiplying by *x* and integrating in *n* steps leads us to the equality

$$P_n(x)\sin x + Q_n(x)\cos x = \frac{x^{2n+1}\cdot 1!!}{(2n+1)!!\cdot 1!} - \frac{x^{2n+3}\cdot 3!!}{(2n+3)!!\cdot 3!} + \frac{x^{2n+5}\cdot 5!!}{(2n+5)!!\cdot 5!} - \dots,$$

where $P_n(x)$ and $Q_n(x)$ are some polynomials of degree not exceeding n with integral coefficients.

Take $x = m$ in equality (1). Then

$$Q_n(m) = \frac{m^{2n+1}\cdot 1!!}{(2n+1)!!\cdot 1!} - \frac{m^{2n+3}\cdot 3!!}{(2n+3)!!\cdot 3!} + \frac{m^{2n+5}\cdot 5!!}{(2n+5)!!\cdot 5!} - \dots, \qquad (1)$$

and on the right-hand side we have a series that is majorized by

$$\frac{m^{2n+1}}{(2n+1)!!} + \frac{m^{2n+3}}{(2n+3)!!} + \frac{m^{2n+5}}{(2n+5)!!} + \dots,$$

and consequently it converges. For large n, series (1) becomes of Leibniz type, and its sum will be positive. Finally, choosing n sufficiently large, it can be made arbitrarily small, since, in absolute value, it does not exceed the nth remainder of the converging series

$$\sum_{k=1}^{\infty} \frac{m^{2k+1}}{(2k+1)!!}.$$

Thus, the right-hand side of equality (1) occurs between 0 and 1, while its left-hand side remains integral. The contradiction obtained proves the theorem.

It turns out that this proof can be remade as school-elementary one. For this you have to change Taylor series by elementary "Taylor inequalities" and integrals of simple functions by its primitives.

For many times we shall use the following obvious fact.

Lemma. *Let* $F(x)$ *and* $G(x)$ *be primitives of the functions* $f(x)$ *and* $g(x)$*, respectively, that acquire the value* 0 *at zero point*:

$$F(0) = G(0) = 0.$$

If for $x \ge 0$

$$f(x) \geq g(x),$$

then

$$F(x) \geq G(x).$$

Proof. The function $F(x)-G(x)$ increases as $x \geq 0$, because its derivative $F'(x)-G'(x)=f(x)-g(x)\geq 0$. Since $F(0)-G(0)=0$, we have $F(x)-G(x)\geq 0$, as $x \geq 0$. The proof of the lemma is complete.

School-elementary proof

Let us now begin with an evident inequality $(x \geq 0)$

$$-1 \leq \cos x \leq 1.$$

We write inequalities comprised of the primitives:

$$-x \leq \sin x \leq x,$$
$$-\frac{x^2}{2} \leq -\cos x + 1 \leq \frac{x^2}{2},$$
$$-\frac{x^3}{6} \leq -\sin x + x \leq \frac{x^3}{6}.$$

We rewrite the last inequality as follows:

$$x - \frac{x^3}{6} \leq \sin x \leq x + \frac{x^3}{6}.$$

Even a weaker inequality

$$x - \frac{x^3}{3} \leq \sin x \leq x + \frac{x^3}{3}$$

will be sufficient. Let us multiply it by x,

$$x^2 - \frac{x^4}{3} \leq x \sin x \leq x^2 + \frac{x^4}{3},$$

and only now take the primitives equal to 0 at zero. It is trivial to guess the primitive of the function $x \sin x$: $-x\cos x$ comes forward, its derivative is equal to $x \sin x - \cos x$, therefore the primitive of the function $x \sin x$ is equal to $-x \cos x + \sin x$. We obtain the inequality

$$\frac{x^3}{3} - \frac{x^5}{3\cdot 5} \le -x\cos x + \sin x \le \frac{x^3}{3} + \frac{x^5}{3\cdot 5}.$$

Further proceeding we multiply by x,

$$\frac{x^4}{3} - \frac{x^6}{3\cdot 5} \le -x^2\cos x + x\sin x \le \frac{x^4}{3} + \frac{x^6}{3\cdot 5},$$

and write inequalities of the primitives (it is also easy to find the primitive of the function $-x^2\cos x$: since $(x^2\sin x)' = 2x\sin x + x^2\cos x$, the primitive of $x^2\cos x$ is $x^2\sin x$ minus the known primitive). Consequently the primitive of $-x^2\cos x + x\sin x$ is $P_2(x)\sin x + Q_2(x)\cos x$, where $P_2(x)$ and $Q_2(x)$ are certain polynomials with integer coefficients. We have

$$\frac{x^5}{3\cdot 5} - \frac{x^7}{3\cdot 5\cdot 7} \le P_2(x)\sin x + Q_2(x)\cos x \le \frac{x^5}{3\cdot 5} + \frac{x^7}{3\cdot 5\cdot 7}.$$

Let us proceed further multiplying by x and taking primitives. Obviously, after n steps (n is any positive integer), we obtain

$$\frac{x^{2n+1}}{(2n+1)!!} - \frac{x^{2n+3}}{(2n+3)!!} \le P_n(x)\sin x + Q_n(x)\cos x \le \frac{x^{2n+1}}{(2n+1)!!} + \frac{x^{2n+3}}{(2n+3)!!},$$

where $P_n(x)$ and $Q_n(x)$ are certain polynomials with integer coefficients, and $(2k+1)!! = 1\cdot 3\cdot 5\cdot \ldots \cdot (2k-1)\cdot(2k+1)$. Namely this inequality is a clue to the proof of the irrationality π.

Assume on the contrary, that π is rational. Thus, $\pi = \frac{p}{q} = \frac{2p}{2q} = \frac{m}{2q}$, where p and q have no common factors, so m and q are certain fixed positive integers. Then $\cos 2q\pi = \cos m = 1$, $\sin m = 0$. Let us write the last inequality namely for this m, i.e., take $x = m$. By inserting $\cos m$ and $\sin m$ values (1 and 0 respectively), we obtain

$$\frac{m^{2n+1}}{(2n+1)!!} - \frac{m^{2n+3}}{(2n+3)!!} \le Q_n(m) \le \frac{m^{2n+1}}{(2n+1)!!} + \frac{m^{2n+3}}{(2n+3)!!}.$$

Recall that *m* is a constant number, while *n* can be chosen free. Let us rewrite inequality (2) as

$$\frac{m^{2n+1}}{(2n+1)!!}\left(1 - \frac{m^2}{2n+3}\right) \le Q_n(m) \le \frac{m^{2n+1}}{(2n+1)!!}\left(1 + \frac{m^2}{2n+3}\right).$$

Since $(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)$ has more than $\left[\frac{n}{2}\right]$ multipliers greater than *n*, we have $(2n+1)!! > n^{n/2}$, and assuming $n > 2m^6$, we derive

$$\frac{m^{2n+1}}{(2n+1)!!} < \frac{m^{3n}}{n^{n/2}} = \left(\frac{m^6}{n}\right)^{n/2} \le \frac{m^6}{2m^6} = \frac{1}{2},$$

$$\frac{m^2}{2n+3} < \frac{m^2}{4m^6+3} < 1.$$

Consequently,

$$0 < Q_n(m) < \frac{1}{2}.$$

However, the coefficients of polynomial $Q_n(x)$ are integral, therefore $Q_n(m)$ is an integer between 0 and $\frac{1}{2}$. That is a contradiction.

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WHEN THE *n*-TH ROOT OF A SUM OF TWO SQUARES IS A SUM OF TWO SQUARES

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Abstract

When is $c = \sqrt[n]{a^2 + b^2}$ *a sum of two squares if* c, a, b *are integers? Using some elementary results from number theory about sum of two squares we get the answer to this question.*

Pythagorean Theorem is perhaps the most known mathematical result. But many people don't know about interesting consequences of this theorem about Pythagorean triple.

Notation. A triple of positive integers (a, b, c) is said to be a Pythagorean triple if it satisfies the Pythagorean equation $a^2 + b^2 = c^2$. If a, b and c have no common factor (i.e. if any pair is co-prime) then this triple is said to be a primitive Pythagorean triple.

It is well-known (Edwards, 2000; Ribenboim, 1999) that a triple (a,b,c) of integers is a primitive Pythagorean triple if and only if it may be written in the form $(2uv, u^2 - v^2, u^2 + v^2)$ or $(u^2 - v^2, 2uv, u^2 + v^2)$, where u, v are relatively prime positive integers of opposite parity. Therefore, we can see that integer $c = \sqrt{a^2 + b^2}$ is representable as sum of two squares $c = u^2 + v^2$.

The results about primitive Pythagorean triple were known to Euclid, Pythagoras, and Plato (Ribenboim, 1999) and the proof is elementary. Using elementary well-known results from number theory about the sum of two integers (Edwards, 2000; Mačys, 2007; Ribenboim,1999) we prove Theorem 1. The proof of Theorem 1 is also elementary. We prove some elementary facts of the theory of numbers in the convenient form which is useful to us.

It is not difficult to verify formula

$$(a^2+b^2)(c^2+d^2)=(ac\mp bd)^2+(ad\pm bc)^2. \qquad (1)$$

Therefore, the product of two numbers is representable as sum of two squares.

We denote by $\gcd(a,b)$ the greatest common divisor of numbers a and b. We assume that components in a representation of the number by a quadratic form are always integers, for instance, the number $n=a^2+b^2$ has integer components a and b.

Proposition 1. If number a^2+b^2, where $\gcd(a,b)=1$, is divisible by the prime number p, then p is representable as sum of two squares, i.e. there exist u and v such that $p=u^2+v^2$.

For an elementary proof, see (Edwards, 2000; Mačys, 2007; Ribenboim, 1999). We note that from $\gcd(a,b)=1$ and $a^2+b^2\neq 1$ it follows that $a\neq 0$ and $b\neq 0$.

Proposition 2. If number a^2+b^2 is divisible by the prime number p which is representable as sum of two squares (there exist u and v such that $p=u^2+v^2$), then

$$a^2+b^2=(u^2+v^2)(x^2+y^2)$$

for some numbers x and y.

Proof. The product

$$(au+bv)(au-bv)=a^2u^2-b^2u^2=a^2u^2+a^2v^2-a^2v^2-b^2v^2=$$
$$a^2(u^2+v^2)-v^2(a^2+b^2)=a^2p-v^2(a^2+b^2)$$

is divisible by p. Since p is prime, $au + bv$ or $au - bv$ is divisible by it. Applying formula (1) we obtain

$$\frac{a^2+b^2}{u^2+v^2} = \frac{(a^2+b^2)(u^2+v^2)}{(u^2+v^2)^2} = \frac{(au \mp bv)^2 + (au \pm bv)^2}{(u^2+v^2)^2} =$$
$$= \left(\frac{au \mp bv}{u^2+v^2}\right)^2 + \left(\frac{au \pm bv}{u^2+v^2}\right)^2 = \left(\frac{au \mp bv}{p}\right)^2 + \left(\frac{au \pm bv}{p}\right)^2. \qquad (2)$$

Consequently, for appropriate choice of sign in the brackets at the end of (2) the above number $x = (au \mp bv)/p$ and $y = (au \pm bv)/p$ are integers. From (2) it follows that $a^2 + b^2 = (u^2 + v^2)(x^2 + y^2)$.

Proposition 3. If each prime factor of number $a^2 + b^2 \neq 0$ is not representable as sum of two squares, then $a = 0$ or $b = 0$.

Proof. In the opposite case we have $a \neq 0$ and $b \neq 0$. If $m = \gcd(a,b)$ then $a^2 + b^2 = m^2(c^2 + d^2)$, where $1 = \gcd(c,d)$. By virtue of Proposition 1 the number $c^2 + d^2$ has at least one prime factors which is representable as sum of two squares. We have an absurd.

Proposition 4. Suppose that $a = c^n$, $n \in \mathrm{N}$, has a prime decomposition into an infinite number of factors

$$a = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k},$$

then each a_i, $i = 1...k$, is divisible by n.

Proof. It is clear that each prime factors of the number c is prime factors of the number a and contrary otherwise. This yields that $c = p_1^{b_1} p_2^{b_2} ... p_k^{b_k}$. Then

$$c^n = p_1^{nb_1} p_2^{nb_2} ... p_k^{nb_k} = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}.$$

By Fundamental Theorem of Arithmetic this decomposition is unique, therefore we obtain $n\beta_i = \alpha_i$, $i = 1...k$.

Proposition 5. If $ab = c^n$, where $\gcd(a,b) = 1$, then a and b are nth powers; that is, there exist positive integers for which $a = c_1^n$ and $a = c_2^n$.

For a proof, see (Burton,2007; Edwards, 2000; or Postnikov, 1982). However, this Proposition is not difficult to proof using the Proposition4.

Suppose $c^n = a^2 + b^2$. When $c = \sqrt{a^2 + b^2}$ is a sum of two squares if c, a and b are integers? It is not true always. For instance, we have $15^2 = 3^2(2^2 + 1^2)^2 = 9^2 + 12^2$, i.e. 15^2 representable as sum of two squares. Thus we have $15 = 3(2^2 + 1^2)$. Assume that 15 are representable as sum of two squares. If we divide 15 by $5 = 2^2 + 1^2$ which is representable as sum of two squares, then we obtain 3. By virtue of Proposition 2 the number 3 must be representable as sum of two squares. However, 3 has form $4k + 3$, then from the famous Fermat's Theorem (Burton, 2007) it follows that 3 is not representable as sum of two squares. We have a contradiction.

It is not difficult to prove that 3 cannot be representable as sum of two squares don't using Fermat's Theorem. In fact, let's assume the opposite $3 = a^2 + b^2$, then $0 \le a^2 \le 3$ and $0 \le b^2 \le 3$, whence $0 \le a \le 1$ and $0 \le b \le 1$. Changing the value for $a = 0, 1$ and $b = 0, 1$, we see that $3 \ne a^2 + b^2$.

Theorem 1. Suppose that $c^n = a^2 + b^2$, where a, b and c are integers and $m = \gcd(a,b)$. Let $m = m_1 m_2$, where m_1 has prime factors q_i, $i = 1...k$, which are not representable as sum of two squares and m_2 has prime factors p_j, $j = 1...l$, which are representable as sum of two squares. Then $c = \sqrt[n]{a^2 + b^2}$ is representable as sum of two squares if and only if number $e = \sqrt[n]{m_1}$ is an integer.

Proof of Theorem 1. Sufficiency. If $m = \gcd(a,b)$and $m = m_1 m_2$ as in the Theorem, then

$$a^2 + b^2 = m^2\left(a^2 + b^2\right) = m_1^2 m_2^2\left(x^2 + y^2\right),\ gdc(x,y) = 1.$$

By the assumption numbers m_1^2 and m_2^2 don't have common factors. Each prime factors of $x^2 + y^2$ is representable as sum of two squares by virtue of Proposition 1. So, m_1^2 *and* $m_2^2(x^2 + y^2)$ are relatively prime. Consequently, by virtue of Proposition 5 we obtain

$$c_1^n = m_1^2, c_2^n = m_2^2\left(x^2 + y^2\right).$$

Each prime factor of m_2 is representable as sum of two squares by assumption of Theorem. This yields that $c_2^n = m_2^2(x^2 + y^2)$ has a prime decomposition $c_2^n = m_2^2(x^2 + y^2) = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ into a finite number of factors $p_i = u_i^2 + v_i^2$, $i = 1...k.$,and each α_i, $i = 1...k$, is divisible by n by virtue of Proposition 4, i.e. $n\beta_i = \alpha_i$. Hence, we obtain

$$\begin{aligned} c^n &= p_1^{a_1} p_2^{a_2} ... p_k^{a_k} = p_1^{n\beta_1} p_2^{n\beta_2} ... p_k^{n\beta_k} = \\ &\left(u_1^2 + v_1^2\right)^{n\beta_1}\left(u_2^2 + v_2^2\right)^{n\beta_2} ... \left(u_k^2 + v_k^2\right)^{n\beta_k} = \\ &\left(\left(u_1^2 + v_1^2\right)^{\beta_1}\left(u_2^2 + v_2^2\right)^{\beta_2} ... \left(u_k^2 + v_k^2\right)^{\beta_k}\right)^n. \end{aligned} \tag{3}$$

By formula (1) we multiple all numbers as sum of two squares in the brackets at the end of (3). This yields that $c_2^n = \left(f^2 + g^2\right)^n$ and thus

$$c^n = c_1^n c_2^n = m_1^2\left(f^2 + g^2\right)^n. \tag{4}$$

Whence, if $e = \sqrt[n]{m_1}$ is an integer, then

$$c = \sqrt[n]{m^2}\left(f^2 + g^2\right) = \left(\sqrt[n]{m}\right)^2\left(f^2 + g^2\right) = e^2\left(f^2 + g^2\right) = (ef)^2 + (eg)^2.$$

Necessity. From (4) we have $c = \sqrt[n]{m_1^2}\left(f^2 + g^2\right)$. If the number c is representable as sum of two squares, then we can divide c by $f^2 + g^2$. By virtue of Proposition 2 the number $c/\left(f^2 + g^2\right)$ is an integer and it must be representable as sum of two squares. We denote $c/\left(f^2 + g^2\right) = s^2 + t^2$, then from (4) we obtain

$$\left(s^2 + t^2\right)^n = m_1^2. \tag{5}$$

Consequently, $s^2 + t^2 = \sqrt[n]{m_1^2} = \left(\sqrt[n]{m_1}\right)^2$. From (5) it is follows that each prime factor of the number m_1 is the prime factor of the number $s^2 + t^2$. By assumption of Theorem 1 all prime factors of the number m_1 are not representable as sum of two squares. Then by virtue of Proposition 3 we have $s = 0$ or $t = 0$. Therefore, we obtain $s^2 = \left(\sqrt[n]{m_1}\right)^2$ or $t^2 = \left(\sqrt[n]{m_1}\right)^2$. This yields that number $\sqrt[n]{m_1}$ must be integer.

Theorem 2. Suppose that $c^n = a^2 + b^2$, where a, b and c are integers and $m = \gcd(a,b)$. Then $c = \sqrt[n]{a^2 + b^2}$ is representable as sum of two squares if number $e = \sqrt[n]{m}$ is an integer.

Proof is evident and it follows from Theorem 1.

Theorem 3. Suppose that $c^n = a^2 + b^2$, where a, b and c are integers and $1 = \gcd(a,b)$. Then $c = \sqrt[n]{a^2 + b^2}$ is always representable as sum of two squares.

Proof is evident and it follows from Theorem 1.

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CONTENT AND METHODOLOGY: DIVERSE CHOICE OPPORTUNITIES IN THE TOPIC "MATHEMATICAL STATEMENTS AND PROOFS"

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Abstract

This paper deals with the content of methodology material and the results of its approbation. The methodology material is developed for grade 10 on the theme "Mathematical statements and proofs".

The new secondary education standard came into force in Latvia in September 1, 2008. The program of mathematics for secondary school was also changed, the new topic in an amount of 18 study- hours – „Mathematical statements and proofs" was introduced. Therefore a goal has been set up – to prepare methodological material on this new topic.

This methodological material contains:

- a thematic plan for the topic;
- 19 lesson plans;
- 14 worksheets for students;
- 9 worksheets for teachers;
- 8 presentations.

The planning of the topic „ Mathematical statements and proofs"

The topic was designed into 4 modules and two summary lessons.

1. Introduction to the theory of sets (3 lessons) – students get acquainted and acquire the basic concepts of the theory of sets, and operate with the sets;

2. Statements and adjudications (5 lessons) – students form statements, define their authentication and carry out adjudications; students differentiate common and segregated expressions, determine the authentication of the expressions, as well as can create converse examples;
3. Definitions, axioms and theorems in mathematics (2 lessons) – students know how to find imperfections in definitions and correct them; students get acquainted with the concepts: axiom, theorem, quality and indication, and the differences among these concepts.
4. Proofs (7 lessons) – students get acquainted with different ways of proofs: direct proof, proof from the converse and the principle of the mathematical induction; they can differentiate and use them.

Examples of tasks in the theme

The tasks are grouped according to modules.

1. Introduction to the theory of sets

   1.1. There are 630 students in a school. All of them study English. 380 of them study German as well, but only 45 of them study French. 24 students study 3 languages. Plot the sets, using the Eiler – Venn diagram. (E – students who study English, D – students, who study German, F – students, who study French). Define number of elements of the following sets: $E \bigcup D$; $F \setminus E$; $D \bigcap F$; $E \bigcap D$; $D \setminus F$!

   1.2. The sets A, B and C have some common elements, but the set D has only common elements with the set A. The sets A, B, C have no common elements with the set C. The sets A, B, C, D, and E are subsets of the set F. Draw an appropriate Euler – Venn's diagram!

2. Statements and adjudications

   2.1. Is the statement – a number in form $2n+1$ divides by 5 if $n \in Z$, true or false? If false, give a converse example.

   2.2. Invent two assertions which are not statements, and write them down!

   2.3. Statement A – number 8 is a full number, statement B – number 8 is a natural number, statement C – number 8,1 is a full number. Create statements „A or B" and „A and C", define their authenticities!

3. Definition, axiom and theorem

   3.1. A following definition of a parallelogram is given – perpendicular is the height of the parallelogram. Is this kind of definition proper? If not, why?

   3.2. Statements are given – alternate angles are equal; only one straight line can be drawn through any two points in a space. Which one of these statements is an axiom and which one is a theorem? Why? What is the condition and what is the induction of the theorem?

4. Proofs

   4.1. If n is an even number, then $2n+2$ is also an even number – prove it by using the direct proof!

   4.2. Two straight lines that are perpendicular to the third line are mutually parallel. Prove it by making proof from the converse!

Proposed teaching methods

Whilst planning the methodological material, the attention was paid not only to the mathematical content, but also to the most suitable methods of effective presentation of the content and to the study goals. The following methods were predominantly chosen in order to promote development of students' cognitive skills:

- Cooperative learning (different ways of statements are acquired)
- Work with the text (students get acquainted with the basic concepts of the theory of sets)
- Work in pairs (common and segregated expressions, explanation of concepts)
- Discussions (about the necessity of the theme in secondary school)
- Visualization (web of thoughts)
- Narrative/lecture (about definitions, theorems, sets of numbers)
- Creating exercises (creating exercises according to the Eiler-Venn diagram)
- Demonstration (mathematical induction, sets)
- „Stations" (getting prepared for final test)

Unusual tasks are provided, for example, to create a textual exercise with numerical values for a particular Euler –Venn's diagram.

In order to promote creative thinking and to reinforce comprehension, it is offered to the students to create an advertisement for different concepts: the basic concept, definition, axiom, reverse theorem, converse theorem.

"Station"

1. Station „Statements"

Exercise 1. *Define if the given assertion is a statement.*

a) Each equation in form $ax = b$*, where a, b are real numbers, is a linear equation.*

b) $x > 6$ *is an inequality.*

c) *December is the last month of the year.*

Exercise 2. *A common statement is given: „Each real number a and b has a quality:* $a + b = b + a$ *". Write 3 separate statements! Define authenticities of the common and the separate statements!*

2. Station „The different forms of a statement"

Exercise 1. *Two statements are given: A – function is linear, B – function is an even function. Create statements:*

a) A or B

b) A and B

c) neither A nor B

d) A only if B

Exercise 2. *Define if the statement from the previous exercise d) is true. Find a function to which the statement of the previous exercise would be true.*

3. Station „Judgments"

Exercise 1. *Name examples of deductive and inductive judgments.*

Exercise 2. *In a family triplets were born: Rita, Vita and Gita. In a while they created a custom – if anyone ever asked a question, two of them would tell the truth but the third one would lie. When a woman asked them who the first one to be born was, she received the following answers. Rita: „Vita was the first." Vita: „I am not the oldest." Gita: „Rita is the oldest." Who is the oldest sister?*

4. Station „Sets"

Exercise 1. *It is known that* $A \cap B = \{1\}$ *and* $A \cup B = \{0;1;5;9\}$. *What can the sets A and B be like?*

Exercise 2. *A set* $A = \{4;5;9;13\}$ *is given. Find an appropriate set B to make* $A \cap B = A \cup B$.

5. Station „Setting up hypothesis in an inductive way"

Exercise. *Set up hypothesis, what mathematical expression in the general case will make the given sum equal?* $1 + 2 + 2^2 + 2^3 + \ldots 2^{n-1} = ?$

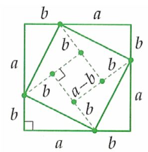
6. Station „Mathematical induction”

Exercise 1. *Name all steps of the mathematical induction!*

Exercise 2. *Prove the equivalence by using the method of mathematical induction* $1\cdot 2+2\cdot 5+3\cdot 8+\ldots+n(3n-1)=n^2(n+1)$*!*

7. Station „Proof using geometrical interpretation”

Exercise. *Write proof of the statement* $(a+b)^2 \geq 4ab$*, using the image below!*



Results of testing

The students’ achievement test composed by the group of experienced teachers on the basis of official study programme was used. I was not acquainted with the test while developing the methodological material. So the methodology offered was not influenced by the type of tasks used in the test. By using this test, I wanted to find out what the results would be like if the methodological material is used in the study process.

The test was taken by the students of two classes. These classes had studied the theme „Mathematical statements and proofs” on the basis of the developed new methodological material. The coefficient of acquirement for the first class was 75%, and for the second class – 63%. One may state that the coefficient of acquirement comparing to average proved to be appropriate in case of both classes.

Students’ opinions about this theme:

- „students prefer solving typical exercises";
- „time planned for the theme could be used more rational";
- „the most capable students already know almost all content of the theme";
- „it is necessary to think logically within the theme (not everyone manages to)";
- „probably this theme helps arranging thoughts".

Possible risk factors

Whilst developing and approbating the methodological material, the following risks were ascertained:

- Professional readiness of the teachers to teach the theme „Mathematical statements and proofs";
- Quality and accessibility of the study materials (now there is only one text book for secondary schools available that includes the theme "Mathematical statements and proofs");
- Students' abilities and prior training (stable knowledge about already acquired concepts in mathematical course is necessary for acquisition of this theme).

Suggestions

Suggestions declared during the work process:

- Proofs should be integrated into the whole mathematical course in secondary school;
- It is necessary to work out a theoretical material for teachers about the theme "Mathematical statements and proofs".

Conclusion

The developed methodological material is recognized as good one. With a good methodological material and a professional teacher the theme „Mathematical statements and proofs" can be acquired in secondary schools. At present this methodological material is the first supporting material in Latvia which reflects all the achievable results within the theme „Mathematical statements and proofs". The theme is necessary for students because it helps to develop and strengthen comprehension about the theory of elements of sets, mathematical concepts and their role in mathematics, judgments, statements and ways of proofs. The theme gives only a notion about the ways of proofs and possibilities of their usage. The proofs should be integrated in the whole mathematical course in secondary school. It is advisable to work out a theoretical material in Latvian for teachers so that the teachers who teach in secondary schools could refresh their knowledge about the theory of elements of sets, mathematical concepts, judgments, statements and ways of proving.

ABOUT PROBLEMS ARISING WHEN GRAPHS OF FUNCTIONS ARE CREATED USING SOFTWARE PACKAGES

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Abstract

In this paper we consider failures arising in constructing graphs if Maple or Mathematica is used. We consider several functions in the form $y = f(x)$ *or* $f(x; y) = 0$ *and show four groups of failures.*

Introduction

It is known that while solving several scientific problems or just conceiving properties of several functions it is useful to construct graphs of these functions. Therefore many people including pupils, students and teachers use software packages. However, not always these packages draw the graphs correctly. The distortion of graphs as well as incorrect printings of numerical values can occur.

Here we consider graphs that are drawn using Maple V release 4 (in one case we use Maple 11, see Figure 4) and Mathematica 2.2.

Groups of failures

The main groups of failures found by author are as follows:

1. **Program package has a regular error when drawing graphs of functions.** For example, graphs of all discontinuous functions are drawn continuously.
2. **Some part of the graph differs substantially from what is expected.** The difference often can be explained mathematically.

3. **Graph of a function is not drawn for all necessary values of argument.** E. g., entering the function, which is defined for all x from interval $[a; b]$, we can obtain the graph only for $x \in [a; b]/I$, where I is some interval.
4. **Non understandable or unreadable numbers are printed near coordinate axis.** It happens mostly in cases when the function is given in an interval with endpoints in decimal form.

First group of failures

Let us consider first group of failures. Author has noticed that drawing graphs of discontinuous functions both Maple and Mathematica draw them as continuous curves. More precisely, suppose that function $y = f(x)$ has such a point of discontinuity x_0 that there exist real y_1 and y_2, $y_1 \neq y_2$, $y_1 = \lim\limits_{x \to x_0 - 0} f(x)$ and $y_2 = \lim\limits_{x \to x_0 + 0} f(x)$. Then graph of $f(x)$ in Cartesian coordinate system will be drawn together with the line segment with endpoints $(x_0; y_1)$ and $(x_0; y_2)$. Let us consider some examples (to construct examples it is appropriate to use discontinuous functions $\{x\}$ and $[x]$).

Example with Maple: plot(frac(x), x=0..4);

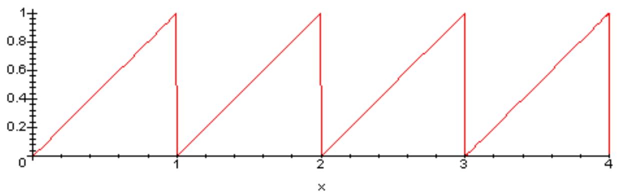

Figure 1

Remark. The following option plot(frac(x), x=0..4, discont=true); gives mathematically correct graph.

Example with Mathematica: Plot[Floor[x], {x, 0, 7}], see Figure 2.

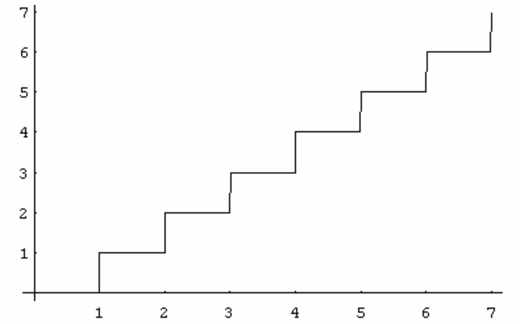

Figure 2

Failures of this type we observe also in polar coordinates. Example with Maple: polarplot([3+frac(3*(t+4)/Pi), t, t=0..2*Pi]);

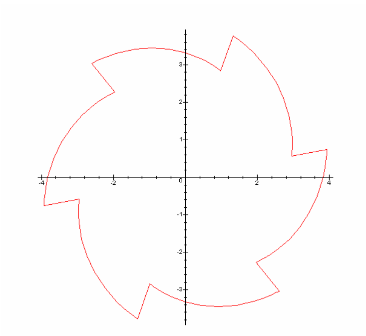

Figure 3

Second group of failures

Now let us consider second group of failures – distorted graphs. At first we consider one $y = f(x)$ type function whose graph is drawn in Maple and Mathematica.

Example with Maple 11: $plot\left(\frac{1}{\frac{1}{\sqrt{x^2+1}-x}-2\cdot x}, x=0..5000\right)$

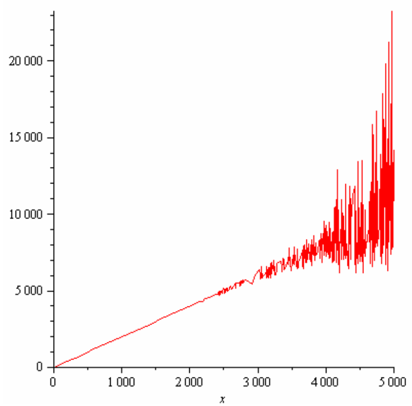

Figure 4

The same function by Mathematica: Plot[1/(1/(Sqrt[x*x+1]-x)-2*x), {x, 0, 5000}], see Figure 5.

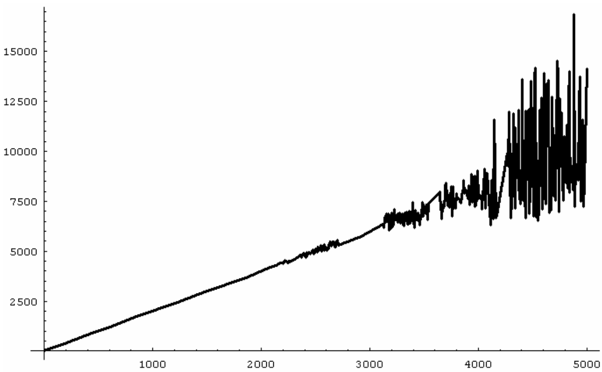

Figure 5

Of course graphs in Figure 4 and Figure 5 are incorrect. Actually function $f(x)$, whose graph is drawn, is equivalent to $\sqrt{x^2+1}+x$. For large x graph of this function is close to a straight line.

Now let us consider $f(x; y) = 0$ type functions.

Example with Maple: implicitplot(x+y=x/y, x=-3..4, y=-1.5..5);

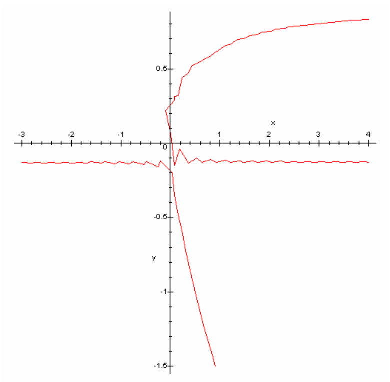

Figure 6

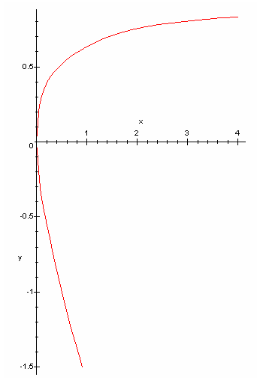

Figure 7

In Figure 6 we see an incorrect graph. One can prove that there must be one branch of hyperbola without point (0; 0) (second branch consist of points $(x; y)$ where $x \leq -4$, so we can not see it). Correct graph is shown in Figure 7.

Remark. However, by implicitplot((x+y)y=x, x=-3..4, y=-1.5..5); we obtain the graph almost the same as in Figure 7.

Next example with Maple: implicitplot(abs(sin(x)-sin(y))=0.2, x=-6..6, y=-6..6);

The graph we see in Figure 8 is incorrect (it follows, for example, from the fact that point $(x_0; y_0)$ can not belong to this graph if $x_0 = y_0$). Correct graph is shown in Figure 9.

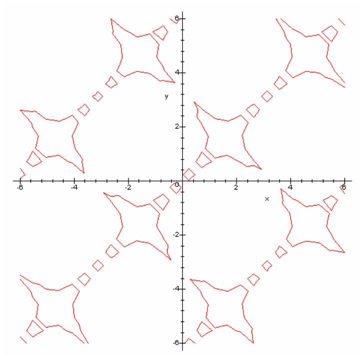

Figure 8

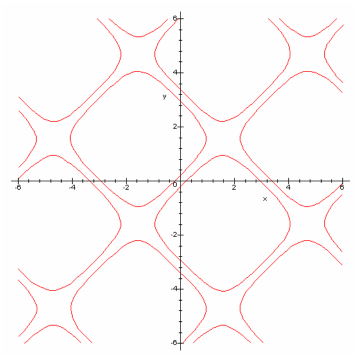

Figure 9

Third group of failures

Now let us consider graphs that are not drawn in an appropriate

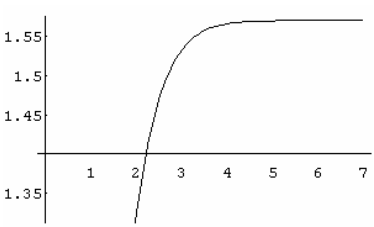

Figure 10

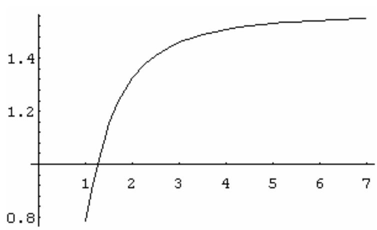

Figure 11

interval. First let us consider example in Mathematica: Plot[ArcTan[x^x], {x, 1, 7}]. We can see the produced graph in Figure 10. It is strange that the graph is drawn in the interval $[a; 7]$ where $a \approx 2$, not in [1; 7]. In Figure 11 we can see the graph obtained by command Plot[ArcTan[x^2], {x, 1, 7}]. This graph is given in an appropriate interval.

Fourth group of failures

Consider two graphs in Figure 12 and Figure 13 which are obtained in Mathematica using commands Plot[x, {x, 1.57, 1.574}] and Plot[x, {x, 1.57, 1.573}].

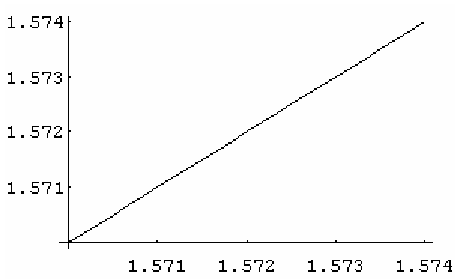

Figure 12

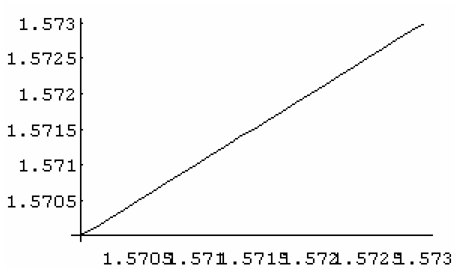

Figure 13

We see that in Figure 13 near the *X* axis numbers are non-understandable because they contain too much digits. In Figure 12 there is no such problem.

Example in Maple: plot(arctan(x)+arctan(1/x), x=0..2);

We can see strange digits near Y axis. The graph also contains unnecessary vertical line segment (see first group of failures).

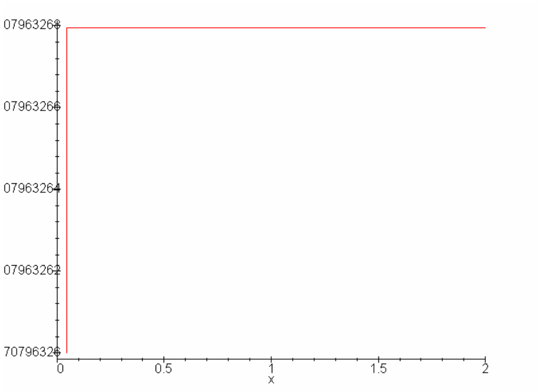

Figure 14

Conclusions

From all above one can conclude that we must be careful if we draw graphs using software packages – sometimes graphs can be incorrect or incomplete. But often mathematical thinking helps us to understand the correct shape. This thinking is useful for all who use software packages.

It would be useful to investigate how do software packages compute the values of mathematical expressions, not only draw the graphs. Very probably there exist expressions that are computed in a false way.

OPERATIONS WITH FRACTIONS IN THE LEARNING ENVIRONMENT T-ALGEBRA

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Abstract

T-algebra is an interactive learning environment for exercises in four subject areas of elementary algebra: integer expressions; fractions; linear equations, inequalities and linear equation systems; monomials and polynomials. The current paper describes operations with fractions.

Introduction

T-algebra is an interactive learning environment for exercises in elementary algebra. It implements 61 task types in four subject areas: integer expressions; fractions; linear equations, inequalities and linear equation systems; monomials and polynomials. Working with T-algebra, the student composes the solution step by step. The program provides feedback and hints, requires correction of all mistakes.

The current version of T-algebra was developed at the University of Tartu from 2004-2008 and the program, together with necessary task files, is available at http://math.ut.ee/T-algebra/. General design ideas of T-algebra, the task solution dialogue and comparison with other programs are published in (Issakova et al, 2006) and also in (Prank et al, 2006, 2006a, 2007). The current paper presents the implementation of operations with fractions in T-algebra. We briefly describe our solution dialog, task types and conversion rules.

The programming of T-algebra was financed by the Estonian Tiger Leap Foundation. Research of the working group is financed by the grant SF0182712s06 of the Estonian Ministry of Education and the Estonian Science Foundation grant ETF7180.

Task solution dialog in T-algebra

Designing the task solution interface of T-algebra, we tried to create an environment where the solution would look similar to written solution, all the decisions and calculations would be made by the student, and the environment would be able to provide hints, to understand mistakes and to give feedback.

Working with T-algebra, the student performs at each step three actions:

1) selects the operation from the menu,
2) marks the operand(s) in expression,
3) enters the result of the operation.

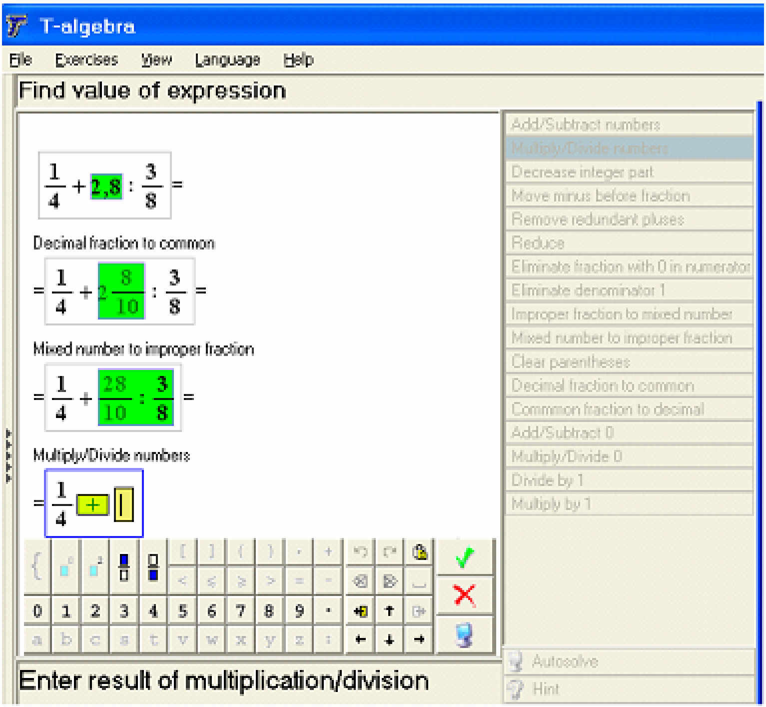

Figure 1: Solution window of T-algebra

Figure 1 presents the solution window of T-algebra. In the first two steps, the student has converted the decimal number 2,8 into mixed number and then to improper fraction. For the third step of the solution, the student has already chosen the operation Multiply/divide numbers and marked two fractions. At the last stage of the step, the student should fill the yellow boxes with components of the result.

T-algebra has three working modes for entering the result of conversion: *free, structured* and *partial*. The input mode is set in the task file and cannot be changed by the student.

In the *free mode*, the student has in most cases only one input box for the result but often also a preceding box for the sign (as in Figure 1). Each box enables entering specific type of content. For instance, the first input box on Figure 1 accepts only a plus or minus sign, while the second box (main part of the result) can contain fractions, numbers and multiplication sign (see the virtual keyboard). The free mode is quite flexible in terms of the form of the result, and the step made by the student can be shorter or longer. For example, in the situation of Figure 1, the results $\frac{28 \cdot 8}{10 \cdot 3}$, $\frac{224}{30}$, $\frac{112}{15}$, $7\frac{14}{30}$, $7\frac{7}{15}$ and several others are acceptable.

The *structured mode* is designed for the cases where the students need more scaffolding (for initial learning of new operations and/or for weaker students). The program offers a structure of input boxes. Figure 2 depicts the input boxes when the above division of fractions is selected in the structured mode:

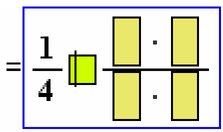

Figure 2: Input phase in the structured mode.

Usually the structured mode of T-algebra allows composing only short solution steps. For example, only the first of the five variants of the result in the above example can be entered in the boxes of Figure 2. Additionally, input in separate boxes enables to generate a more detailed diagnosis of errors.

In the *partial mode*, T-algebra automatically generates outputs of the results that remain unchanged and offers input boxes only for the components that have to be calculated. In most operations with fractions, the structured and partial modes are identical.

Task types for fractions

Task types of T-algebra are taken from the exercises in textbooks. Some types are devoted to one particular operation, while others integrate the knowledge of a whole chapter:

1) Reduce,
2) Extend to denominator …,
3) Add/subtract like fractions,
4) Convert fractions to like fractions,
5) Compare fractions,
6) Mixed number → improper fraction,
7) Improper fraction → mixed number,
8) Add/subtract like mixed numbers,
9) Add/subtract unlike fractions/ mixed numbers,
10) Common fraction → decimal,
11) Decimal fraction → common,
12) Decimal approximation,
13) Multiplication of fractions,
14) Division of fractions,
15) Reciprocal value,
16) Find value of expression with fractions.

The task type of T-algebra determines possible forms of initial expressions, the required form of the answer, and the set of rules available for solution steps. For each task type, T-algebra has a built-in solution algorithm. The student program of T-algebra uses it for finding the right answer, for generating the demo solution, and for giving dynamic hints in any situation. The teacher program additionally uses it for checking whether the entered task can be solved in T-algebra.

Conversion rules (operations)

Figure 3 presents the solution window with a menu of rules when the student solves a task of type Find reciprocal value. The student has pushed the button Solved but the error message appears because the answer does not have the required form.

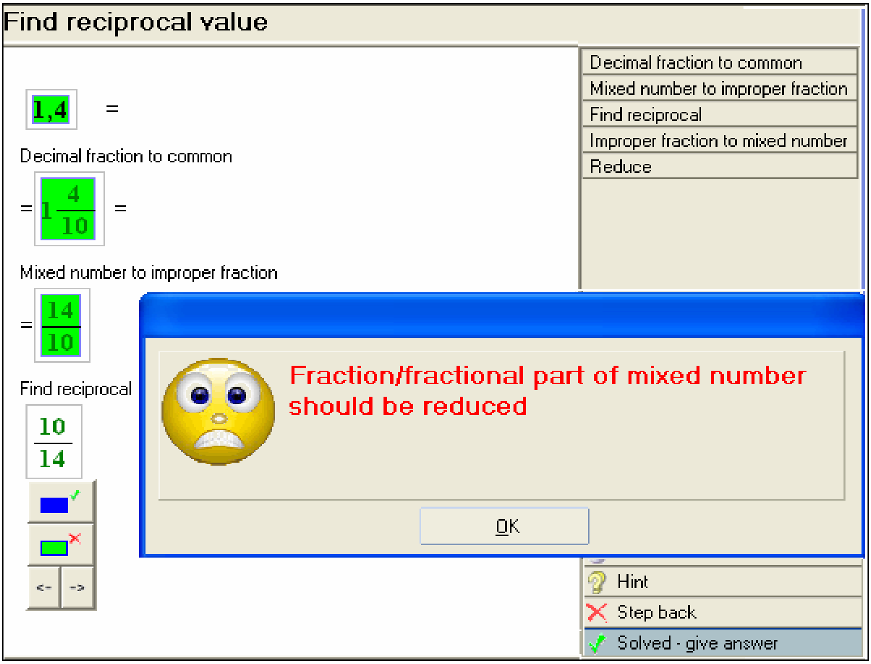

Figure 3: Solution window with rules, solution steps and error message.

We have implemented and included in the menu for each problem type the operations from the textbook algorithm for this problem type, and have also added the necessary common rules for making different auxiliary conversions, calculations and simplifications. For instance, in the above example, it is natural to expect that it is possible to present the given number as a common fraction (proper or improper) before finding the reciprocal value. After the main operation is completed, it should be possible to simplify the result.

Our task types and their solution algorithms immediately necessitated the inclusion of the following rules:

1) Reduce,
2) Extend common fraction,
3) Compare fractions,
4) Add/Subtract,
5) Multiply/Divide,
6) Raise to power,
7) Common fraction to decimal,
8) Decimal fraction to common,
9) Improper fraction to mixed number,
10) Decrease integer part,
11) Mixed number to improper fraction,
12) Mixed number to sum,
13) Common fraction to division,
14) Find appropriate decimal approximation,
15) Round,
16) Find reciprocal.

Some additional fraction-related operations are necessary for simplifications:

17) Move minus before fraction,
18) Eliminate division from fraction,
19) Eliminate fraction with 0 in numerator,
20) Eliminate denominator 1.

The dialog of most of the operations is implemented in our general three-stage style. An important exception is the addition and subtraction of fractions. When the fractions have different denominators, the common denominator should be found and the numerators should be multiplied by corresponding factors before addition. We have implemented all this in an explicit way in the structured mode. The student has to enter first the common denominator, after that the factors, and finally the result.

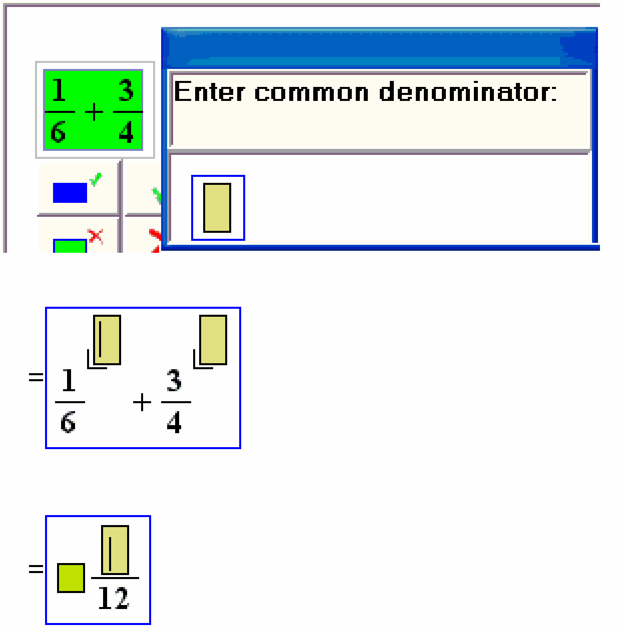

Figure 4: Three consecutive stages of addition of fractions in the structured mode

In the free mode, we have left the preparatory part of this work implicit. The student is simply asked to enter the result.

Figure 5: Addition of fractions in the free mode

In both modes, the student has control over the length of step. He/she can enter the final answer $\frac{11}{12}$ or present the sum as one fraction $\frac{2+9}{12}$; in the free mode, he/she could also only convert addends to common denominator: $\frac{2}{12}+\frac{9}{12}$. The teacher can here recommend an appropriate length of the step.

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MATH CLUB'S EDUCATIONAL SPACE MODELING

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Abstract

This paper is about the educational activity of a math club. This activity is organized by means of educational space modeling. The modeling is based upon integration of intellectual games, comic problems and mathematics. Because the substantive and the informative environment are constantly expanded, the students have to get assess and adapt to the information received during the classes. Such skills are developed in the process of creative problem solving and by means of intellectual games. The combination of entertainment and educational activity enhances the learning process and improves the quality of knowledge received.

Introduction

Games are a valuable and topical means of neogenesis for pupils in secondary school. The level of development of pupils in this age is quite high. They are already able to carry out not only simple cogitative and mnemonic operations, but also to analyze new facts, to build proofs logically correctly, etc. At the same time high degree of emotionality, developed imagination and predisposition to game activities are characteristic for teenagers.

It is possible to allocate the psycho-pedagogical possibilities of game in the course of teaching and learning:

- game is a powerful stimulus for both versatile and strong motivation,
- in game all mental processes become more active, it allows to unite harmoniously the emotional and the rational,
- game promotes involving everyone in active work,

- correctly organized games allow involving the energy which schoolboys spend on out-of-school game activity into the educational process. This is considered to be the purpose of the games at school.

One "fundamental principle of human behavior is that emotions energize and organize perception, thinking and action" (Izard, 1991).

Game allows introducing interest and fascination in the educational process, to make this process creative and entertaining. Besides, the game world emotionally paints educational activity, thereby, making mental processes and functions of the child active.

Psychological needs that are often emphasized in educational settings are autonomy, competence and social belonging (e.g. Boekaerts, 1999). These can all be met in a classroom that emphasizes exploration, understanding and communication instead of rules, routines and rote learning. Mirth is the most desired emotion, but it is rather a by-product of activities and conditions, than a result of the desire to experience it. The state of mirth is connected with the sense of confidence and significance.

In this paper, we will present a teaching approach that is being built around math problems which are for the pupil at the same time cheerful (entertaining, funny, cool) and challenging (difficult). We call this approach “Inspiring Mathematics” (IM).

Theoretical framework

IM is based on three educational approaches: acknowledging affect in math learning (Hannula, 2006), using humour in teaching (Grecu, 2008) and use of open problems in math teaching (Pehkonen, 2004).

The educational space represents a set of conditions, influencing educational activity. The given space is constructed in such a manner that teamwork of the teacher and pupils accepts dialog character; and a favorable educational atmosphere that influences interest to mathematical educational activity is being created. Designing of educational space is carried out on the base of obligatory combination of the constructs of the inspiring mathematics.

The inspiring mathematics is a method in which:

1. In the same assignment entertainment is combined with a set of difficulty levels and ways of problem solving. It argues that:

a) emotional mechanisms promote localization of a search zone of the directions and the basic principle of the problem's solution;

b) the problems posed cannot be solved via a kind of well-known algorithms that are so characteristic of typically trivial problems.

2. During problem solving there are conditions for emotions to rise, because:

a) affective components excite the interest and concentration of attention;

b) emotional activation precedes problem solving, representing an emotional anticipation of finding a principle of the given problem's solution.

3. The assignment allows all pupils to participate meaningfully regardless of their abilities, because:

a) there can be different results from non-standard approaches to the problem solving;

b) in the problem there are always several questions of different complexity levels;

c) pupils have the opportunity to choose their level of competence.

The humorous content of the problems and entertaining conditions bring a positive emotional component to the educational activity. The club creates the atmosphere of joyful cognition and "opening" of new knowledge. The intellectual games being used (chess, go, katamino, Hanoi Tower, etc) enable students to develop their personal potential. Proper assessment of the position, the accurate calculation of the chess move, finding a beautiful solution, all this is the result of logical thinking. Used in study, the games serve as a form of cognitive activity. The game allows increasing and strengthening involvement in the learning process because of its specific characteristics of the essence of activity, amusing situations and a high degree of emotion.

Research problem and questions

We wish to study what character of educational activity is preferred by schoolchildren, what interaction is necessary during the learning for increasing of interest to a subject, what emotions accompany the process of learning. The following questions are defined:

1. What kind of mathematical problems inspire schoolchildren from their point of view?
2. What mechanisms should be used to raise attractiveness of educational activity?

Results and conclusions

The study was carried out using a mixed method. We used qualitative and quantitative methods. Questionnaires included open and closed questions with multiple choice questions.

The study of the problem, presented in this paper, began in December 2007 in Espoo school and continued in 2008 in Helsinki school. As the respondents were pupils of grades 7-8. In total the survey was attended by 80 persons.

The study consisted of the following phases:

a) In December, 2007 surveyed pupils' preferences of entertaining features in math. The data were collected in two 7th grade classes and one 8th grade class (40 pupils) after the introduction into the educational process of inspiring mathematics within 5 months. The analysis of the questionnaires, allowing to determine priority directions in the enter-

taining mathematics, has shown that the frequencies of choices were as follows:
- tasks of comic character (55%),
- “something else” (27%),
- “cutting and drawing” (25%),
- “Lego and Chess” (15%),
- “unusual names and properties” (12, 5%),
- “fairy-tale story” (10%).

b) The results of this survey have shown (September 2008) that 74% of participants had stated that it was possible to feel pleasure at mathematics lessons; 50% from them had pleasure playing chess, and 29% think of personality of the teacher as the defining factor.

The received experimental data allow us to draw preliminary conclusions.

1. The selections made by pupils have shown that the priority direction of inspiring mathematics is solving of problems of comic character.

2. Usage of intellectual games in educational activity allows creating an atmosphere of pleasure and emotional rise at a lesson.

The introduction into the educational process of such mechanisms as IM constructs, and intellectual games influences the atmosphere in the classroom.

It promotes positive changes of the pupils’ perception of educational activity. Educational process becomes more active, involving all pupils irrespectively of their abilities and preferences into educational activity.

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TEACHING MATHEMATICS: PUZZLES AND FAIRY-TALE PROBLEMS

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Pavel Shmakov, University of Helsinki

Introduction

This article is dedicated to enhancing teaching mathematics at the school. Effective and widely accessible methods of such teaching are presented. Relevance of the implementation of new methods is determined by the continuous reduction of the interest of the students to study mathematics and other sciences. This fact is alarming and is discussed in mathematical society. Historical-philosophical and professional mathematical aspect of this problem is considered in the article by Novikov (2006). In our work we propose a methodological solution of this problem.

From our point of view, the school system is seriously losing the competition with the «aggressive» mass-media. Among these we mention television, cinema (and, in particular, some types of cartoon animation), comic books, Internet and computer games. The development of the school education has always been evolutionary and slow. Multimedia develops in a revolutionary way, in an explosive manner both quantitatively and qualitatively. Also it is worth to note that currently the school in the children eyes is no longer the only most authoritative source of knowledge about the world. He easily draws the knowledge or more precisely «pseudo knowledge» from animated cartoons, comics or computer games. The school often turns to him simply to a place for mandatory, dull and completely pointless pastime. Besides of the described phenomena one should note also the gradual shift of the center of gravity of the entire school system from the precise and natural sciences to the humanitarian side. This «drift» of the education system generates contradictions, e.g. between among the math teacher - «tormentor», and the «humanitarian-oriented» students - «martyrs».

Without taking into account the pressure of that, external in relation to the school, strongest field of information, it is more difficult to build the best line of teaching all the school subjects. In our opinion, the situation especially with mathematics is difficult. Mathematical tasks, from ordinary pupil's point of views, are either completely abstract, or are bored flimsy, primitive and monotonous models of reality.

This situation can to some extent be adjusted by using the methods proposed by the authors of the article.

Applied tools and techniques:

1. *Active use of imagination of the child and the desire for the puzzles and paradoxes.* We propose to use the innate curiosity and imagination of the child, as an important pedagogical resource. "Every parent knows that children are spontaneously curious and that they have a fantasy, and that they like fairy tales and fantastic stories ..." (Toom, 1999). In our view, this is particularly useful in the early stages of learning mathematics. However, this idea remains valid at all levels of schooling. It is also worth to add that, from a formal point of view, the tale is nothing other than the «abstract», a description of the world. Hereinafter, the term of the abstract description actually means its model, reflecting only some significant, in a certain context, properties and relationships of the surrounding real world. We note that the latter characterizes also mathematics.

The difference between stories and mathematics, in the comparison made above, it is quite understandable: the mathematics models the relationships of real objects, expressed as numerical relationships and relationships of geometric figures, while tales, mainly, describe the relationship of human beings. The main idea of our approach is building a bridge between these descriptions of reality, in order to use the corresponding synthetic product in the process of teaching mathematics in elementary and secondary schools. From the foregoing, we completely leave out of the scope of our discussion important problems of the direct teaching standard mathematical algorithms and, in general, direct study of mathematical structures themselves. Of course, our approach can not solve the global problem – in that case a radical reform of education is needed. Further, we note also the possibility of using mathematical problems with the fantastic, mysterious and paradoxical content of overtime work with children, for example, in mathematical circles. This has been successfully tested by the author's articles for some years in several schools in Helsinki.

2. Emotional mobilisation of training activities. One "fundamental principle of human behavior is that emotions energize and organize perception, thinking and action" (Izard, 1991).

Emotional support of training activities contributed to its revival. For the teacher's the number-one-problem (from our point of view) is an ordinary boredom, which breeds monotony, «infinite» and «purposeless» (from the students' point of view) calculations and conversions. For strong students, boring may be too simple (routine) exercises. On the other hand, for weak students, on the contrary, too complicated tasks cause intuitive desire for distraction. Attention is weakened, the arising «vacuum» is filled with irrelevant thoughts and, indeed much of the class falls out of the educational process. Therefore the interest is considered by us as an important means of attracting attention and supporting students. Because of the unusual content of the proposed tasks, the students have positive emotions that promote perception and, finally, improve the quality of education. Fantastic, mysterious, paradoxical form of text mathematical tasks can be either reproductive or creative. In the first case, students solve problems proposed by the teacher, in the second, they are actively involved in the preparation of the formulation and adaptation of tasks to different conditions. Possible effective cycle using both types of tasks during the lesson is schematically depicted in the chart below (Figure 1).

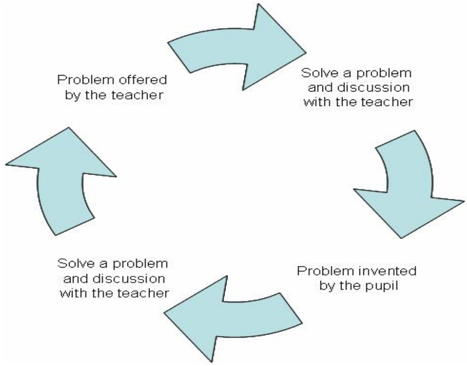

Figure 1: Effective cycle of problems' using during the lesson

At the normal classes or extracurricular activities (hobby groups) one can implement several such cycles.

3. The systematic use of multi-level assignments in the classroom. An additional effect of the techniques that we used can be made with a systematic use of multi-level assignments. Note that almost any task is not difficult to transform into multi-level one. The aim of this approach is to engage to the learning process the maximum number of students, as well as creating the best opportunities for intellectual development. With this approach, in terms of simple tasks designed for all pupils, while tasks of higher difficulty level may be voluntary in nature and dedicated better prepared and talented students. In particular, therefore, we, on the one hand, propose to deal with a possibly stressful situation which often arises in weak and overly humanitarian-oriented students, on the other hand, to raise the intellectual saturation and effectiveness of the learning process.

Another, important task solution, which may contribute to the use of multi-level jobs, is developing a mathematical taste among intellectually capable students.

4. The purposeful use of logic at the mathematical lessons as a method of problem solving (as well as inventing logical problems by the students).

Here we will focus on such an important, but located in the background component of the education of modern man, as logic. The logic is needed, not only to mathematicians, physicists, psychologists and lawyers, investigators, and, of course, politicians, bureaucrats, and just the townsfolk. However, at school there is no such subject. Usually, students are taught logic, copying the logical expressions of teachers and examining the proofs of the theorems in axiomatic theories, but they are known to gradually expel from the school curriculum of most of the developed countries. Thus, in the present circumstances, really one can learn the basics of logic, only when solving specially selected problems. In this context, of course, it may be useful to systematically use mathematical tasks of fantastic, mysterious and paradoxical content. It is also useful to include to the arsenal of educational work also purely logical problem, of course denounced to an appropriate amusing form.

Thus, we distinguish the following reasons for which the use of techniques and methods proposed by us for the educational process is possible and necessary:

- basing on the experience of the child;
- using the natural curiosity of a child;
- using cultural traditions as an important educational potential;
- Encouraging the development of creative potential of students and teachers, as well as the entire class as a social structure;
- contributing to the establishment or strengthening of intellectual and emotional contact between teacher and students;
- significant improvement of the image of a teacher of mathematics, famous among students in their boredom, antiquity and isolation from the real world;

- increasing the competitiveness of the educational system (in the aggregate the information flows surrounding the child);

- establishes a link between the humanities and natural style of learning;

- this approach is more "developing", than just learning computing and study properties of geometric figures from regular students' point of view ;

- develops «flexibility» of students' thinking, logical thinking, the ability to analyze situations and systematically determine the level of its reliability;

- in mathematics lessons elements of applied mathematics and informatics are included;

- there are new possibilities for the study of geometry, and even the topology.

With usual organization form of the lesson, based on our approach, the students develop an ability to competently and productively discuss well-defined problem, and prove their point of view and discuss with the opponent. Let us consider such an important issue as the speech development of the students. This is text problems, especially logical problems, as any other, that contribute to that. Problems of paradoxical, unusual and humorous content allow developing cognitive and creative activity of students. The teachers' "baggage" should include also the classical, well-known problem, which form the examples for reproducing. We point to a few of them: The paradox of Achilles and the tortoise, problem with different combinations of true and false, The use of the exception in solving logical problems, the paradox of a liar. In summary, we will present examples of mathematical problems described above which, in our opinion, are suitable to use in educational process.

Examples of mathematical problems:

"I am this until you do not know what I am. But I am not that when you know that I am. What am I?"

Three tortoises crawl one after another along the road. The tortoise says, "Two tortoises follow my rear". The second says, "One tortoise crawls ahead", "One goes back of me". The third says, "Two are ahead", "One creeps behind". How can this be?"

One should note that this problem is more attractive than something about moving material points along a straight line.

The most common answer here is that it is impossible. But, in fact, there can be different solutions. "3 tortoises crawl…": the words of the third tortoise contradict each other. The solution might be that the last tortoise is lying! …One tortoise is riding on another. …There is a time lapse between the phrases, allowing one tortoise to run ahead. …The fourth tortoise stays near or behind the last turtle, and begins moving after the first phrase of the third turtle… The road is circular… The road is triangular… There is a mirror behind the last turtle. When it looks at its back, it can see one more turtle. Progressing from considered examples, and, instead of tortoise, we turn to another object, e.g., cows. One more possible solution is the birth of a calf!

Let 1 + 1 = 10. Then how many 10 + 11 =?

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THE MEAN VALUE METHOD IN PROBLEMS OF ALGORITHMIC NATURE

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Abstract

New teaching methods and special problems of discrete mathematics have to be implemented in the syllabus of secondary school according to the new educational standards. Good experience can be obtained through the investigation of algorithmic problems. The solution of such problems inspires the mastering of different reasoning methods including the mean value method.

Introduction

The new educational standards in mathematics are defined for the secondary schools in Latvia, including such components as mathematical models, research work and mathematical aspects of interconnection among human, society and environment. The significance of discrete mathematics has been increased substantially. Requirements included in the research activities imply that for students it is worth to acquire the basic knowledge about the algorithms.

The algorithmic problems

Some important classes of algorithmic problems are

problems, where the given finite or infinite process is described by an algorithm and must be analyzed;

problems, where the main task is discovering the proper algorithm;

problems, where the main question is the reachability of particular states of the given process.

It is useful for students to investigate some well known algorithms that can be applied to many problems, for example, Brute force algorithm, Divide and conquer algorithm, Depth – first search algorithm (Aho et al, 1979). The construction of the decision tree can help to examine all the possible cases of some process (Gailītis& Andžāns, 1995).

The mean value method

Analyzing or creating the algorithms the attention is often paid to the following characteristics of the given problem:

the disposition of elements;

the structure which is worth to be set up for execution of algorithm;

the optimal number of iterations;

is the given process finite or not.

In some particular cases above mentioned characteristics can be evaluated by the general methods of reasoning, like **the mean value method** (MVM) (Andžāns et al, 1996):

If a given object is divided into „a small" number of parts, at least one part will be „sufficiently big".

Some popular applications of this method are **Dirichlet's Box principle** (DP), Fubini principle, method of infinite descent, Euclidean algorithm, Sperner's theorem and others. With the help of the MVM one can determine the finiteness of the described process; can evaluate the optimal cardinality of the subsets of given set; can prove the correctness of an algorithm; can estimate the qualitative and quantitative changes of the given structure, etc.

The existence of algorithm

The processing and formalization of given data are important in the investigation of problem. Sometimes the appropriate numerical estimation of them can answer to such relevant question as the existence of the solution.

<u>Problem 1</u>. The house has 100 floors. There are only two switches in the elevator - one goes up 7 floors, another goes down 9 floors (if it is possible). Prove that it is possible to reach every floor!

<u>Comment</u>. The solution of this problem is based on **Bezout's identity**:

If p and q are nonzero integers with greatest common divisor d, then there exist integers m and n such that $pm + qn = d$.

Bezout's identity can be proved by DP in the same way as the Euclidean algorithm (Вагутен, 1972). We can get all possible natural values dk ($d = 1,\ k \in N$) in the case if numbers *p* and *q* are coprime. An appropriate algorithm shows how to go up one floor and how to go down one floor. The alternation of up and down steps helps to not exceed the limits.

The correctness of algorithm

Another step of algorithmic analysis is the verification of the created algorithm. It is essential to check its correspondence to the given demands and to prove its correctness. The feasibility of algorithm's correctness is asserted according to the DP in the solution of following example.

Problem 2. 2*n* guests have attended the party. Every guest has at least *n* friends among the guests. How to place them around the table in such manner that every two neighbours at the table are friends?

Idea of the solution. The given terms can be described by the graph where every guest is represented by a vertex. The edge means the friendship between two guests. This is the Hamiltonian graph according to the **Dirac's theorem**:

A simple graph with n vertices (n ≥ 3) is Hamiltonian if each vertex has degree n/2 or greater.

The goal is to detect the Hamiltonian cycle in the given graph. An idea is to place all vertices arbitrary in the corners of regular polygon. The Hamiltonian cycle is found if every two vertices on the common side of polygon are neighbours. We consider two ordered pairs (A, B) and (C, D) where A and B are non – adjacent vertices, but the edges are (A, C) and (B, D). Then we change the sequence of vertices on the polygon from B to C vice versa (see Figure 1.).

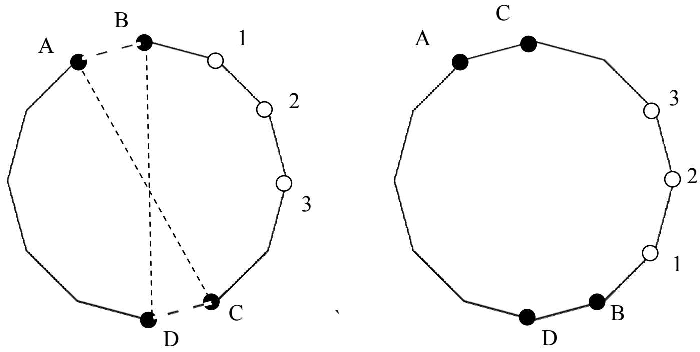

Figure 1: The change of the sequence from B to C.

The mentioned procedure “repairs” the cyclic order of replacement of vertices on the polygon in one or two places (that depends whether the pair (C, D) is an edge or not). Therefore this algorithm is linear and takes less than 2*n* steps. The existence of appropriate couple of pairs in the each step of algorithm can be proved by DP.

The investigation of the given conditions

Not always the real life problems are formulated strictly. The given data can vary. Nevertheless the numerical estimations of them imply the base of the proper algorithm in several cases.

<u>Problem 3.</u> There are 341 stolen diamonds. Each of 22 thieves has a different number of the diamonds (at least 1). The boss calls up any two thieves and collects the difference of diamonds. Than he calls up another two thieves and does the same, etc. Each of the thieves has visited the boss only once. What the boss has to do to collect as much diamonds as possible?

Idea of the solution. We have to estimate the possible sum of all mentioned differences. We place 22 different positive integers in the ascending order $a_1 < a_2 < ... < a_{22}$. There are some numbers smaller and other numbers greater than arithmetic mean value:

$$a = \frac{a_1 + a_2 + ... + a_{22}}{22} = 15{,}5$$

Let us order the given numbers in pairs where the first number is smaller than second in each pair. Let us denote the sum of all first numbers S_1, but the sum of all second numbers S_2. We can evaluate the possible maximal sum of differences in all pairs:

$$(S_2 - S_1)_{\max} = \sum_{i=12}^{22} a_i - \sum_{i=1}^{11} i = 341 - 2\sum_{i=1}^{11} i = 209$$

The worth case of the given sequence is when all the numbers are close to the mean value *a* as much as possible. Then the sequence is

5, 6, 7, 8,... 24, 25, 26

It is easy to generalize the maximal possible sum of the differences in this case is 121 according to the **Proizvolov's identity** (that can be proved by DP):

If the set of numbers {1, 2, 3, ..., 2N - 1, 2N} are split into two sequences, one decreasing and one increasing

$$a_1 > a_2 > a_3 > ... > a_N$$
$$b_1 < b_2 < b_3 < ... < b_N$$

then $$|a_1 - b_1| + |a_2 - b_2| + ... + |a_N - b_N| = N^2.$$

The estimation shows the boss has to align all thieves according to the number of diamonds in ascending order. He can call up one thieve from the beginning of the line but another from the end one pair after another.

The estimation of minimal number of iterations

The time complexity or the number of iterations is relevant criterion for the creation of the sorting or searching algorithms. For example, the weighting problems are the sort of them.

Problem 4. (Moscow Olympiad, 1966) Two of 10 boxes are radioactive. It is possible to find out any radioactivity among the every set of boxes. What is the minimal number of tests to determine the both radioactive boxes?

Comment. The optimal solution of this problem is based on the depth of the decision tree. The bisection method is proper for finding one radioactive box. It is necessary to use n steps by implementing this method for m boxes, where $n = \lceil \log_2 m \rceil$. If we suppose this is sufficient with n steps in the case $m > 2^n$, then in the n^{th} row of the decision tree we have the set with more than one box according to DP, because $\frac{m}{2^n} > 1$.

Similarly the depth of the decision tree depends on the choice of the first control set in the case of two radioactive boxes. The first set must contain such number t of boxes that both branches in the decision tree (for the checking the first and the remaining sets) have approximately equal length. The number t gives the proper result when the expression

$$\left| 2C_{m-t}^2 - C_m^2 \right|$$

is minimal.

The algorithmic processes

Some processes describe the changes that originate in the discrete structures after the iterations. These are problems about the cell division, the moving of objects, the colouring problems. The important questions are the finiteness of process or the numerical estimation of given system in fixed moments.

Problem 5. There are 24 houses in the village painted white or blue. Each owner has an odd number of friends among the other owners. One day the owner mister First decided to paint his house the same colour most of his friends houses had. Mister Second did the same after the month. Then mister Third followed as well as other owners one after another month by month. The process continued – each owner painted his house after every 24 months. Prove there will be the moment that houses do not change the colour.

Comment. The conditions of the given problem are composed on the base of MVM. That guarantees the one part of every owner's friends with buildings in one colour is bigger than the other part. Let us consider the amount of all such pairs of friends with variously painted houses. After one step of the described procedure the number of these pairs does not change or decrease. As the number of pairs is natural it cannot decrease endlessly. The process will become stable. We can mention that this problem uses a kind of the infinite descent method.

Conclusion

There are several reasons why problems of algorithmic nature are fascinating. The students introduce themselves with basic principles of problem investigation. The analysis of well known algorithms demonstrates general methods of problem solution. Due to the continuous evolution of combinatorial algorithms new challenging examples are found in applications to real life problems.

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